Modeling Masonry structures using a non smooth discrete element method. Numerical aspects

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2013
DEM : Context

Masonry as a collection of interacting bodies:
DEM : :Modelling Framework

1. Context
2. Modelling Framework
3. Numerical Strategy
4. Interaction laws relevant for masonry structures
5. Conclusion
DEM : :Modelling Framework : :Motion

Motion can be described through:

- a rigid body motion:
  Generalized coordinates: \( q = \{x_c, R_c\}^T \), where \( x_c = X_c + u_c \) is the position of the inertia center (\( M_c \)) and \( R_c \) the orientation matrix of the body.
  Generalized velocity: \( v^R = \{v_c, \omega\}^T \), where \( v_c \) translation velocity and \( \omega \) angular velocity. Note that: \( \dot{R}_c = \omega R_c^T \)

- a deformation/velocity relatively to the rigid body motion.

The velocity field \( v \) split in a rigid body velocity \( v_R \) and a deformation one \( v_D \):

\[
v(x) = v_c + \omega \times (x - x_c) + v_D(x)
\]

\[
\underbrace{v_R}_{v_R} + v_D(x)
\]
DEM : Modelling Framework : Motion

Velocity field decomposition based on two operators:
- an interpolation operator $G : \mathbf{v}_R = G(\mathbf{v}^R)$
- a projection operator $L$; such that $L(G(\bullet))$ is identity on the image of $G$ mapping, noted $\mathcal{V}_R$

Operators are chosen to split the velocities vector space $\mathcal{V}$ in a direct sum of vector spaces related to pure deformation velocities $\mathcal{V}_D$ and rigid body velocities $\mathcal{V}_R$.

By duality, the forces space $\mathcal{F}$ split in two supplementary spaces $\mathcal{F}_R$ and $\mathcal{F}_D$.

$$\mathcal{V} = \mathcal{V}_R \oplus \mathcal{V}_D \quad <,> \quad \mathcal{F} = \mathcal{F}_R \oplus \mathcal{F}_D$$

$$L \downarrow \uparrow G \quad G^T \downarrow \uparrow L^T$$

$$\mathcal{V}^R \quad <.> \quad \mathcal{F}^R$$
A possible way to build $G$ and $L$ operators is:

\[
G(v^R) = v_R = v_c + \omega \times (x - x_c)
\]

\[
L(v) = \begin{pmatrix}
v_c = m^{-1} \int_{\Omega} \rho v d\Omega \\
\omega = J^{-1} \int_{\Omega} \rho (x - x_c) \times v d\Omega
\end{pmatrix}
\]

where

- mass $m = \int_{\Omega} \rho(x) d\Omega$
- inertia $J = \int_{\Omega} (x - x_c) \cdot (x - x_c) I - (x - x_c) \otimes (x - x_c) \rho(x) d\Omega$
DEM : Modelling Framework : Momentum

The motion of the body is governed by the momentum conservation law:

\[
\text{div}(\sigma(x, t)) + \rho(x, t) f_v(t) = \rho(x, t) \dot{v}(x, t), \forall x \in \Omega(t),
\]

where \( \rho \) is the density of the solid, \( f_v \) the body forces, \( \dot{v} \) the acceleration and \( \sigma \) the stress tensor. Surface forces \( t \) are imposed on \( \Gamma_f \), which normal is \( n (t = \sigma \cdot n) \). Velocities are imposed on \( \Gamma_v \).

Introducing:

- acceleration \( \dot{v} \):
  \[
  \dot{v} = \dot{v}_c + \omega \times (x - x_c) + \omega \times (\omega \times (x - x_c)) + 2\omega \times v_D + \dot{v}_D
  \]

- virtual velocity \( \hat{v}(x) \)
  \[
  \delta V^0 = \{ \hat{v}(x) = \hat{v}_c + \hat{\omega} \times (x - x_c) + \hat{v}_D(x), \forall x \in \Omega(t) \}
  \]
  and homogeneous for fixed degrees of freedom}
DEM : Modelling Framework : Momentum

One obtains:

Newton:

\[ m \ddot{\mathbf{v}}_c + \int_{\Omega(t)} \rho(\mathbf{x}) \dot{\mathbf{v}}_D \, d\Omega = \int_{\Omega(t)} \rho(\mathbf{x}) \mathbf{f}_v(t) \, d\Omega + \int_{\Gamma(t)} \mathbf{t}(\mathbf{x}, t) \, d\Gamma \]

Euler:

\[ \mathbb{J} \dot{\mathbf{w}} + \mathbf{\omega} \times \mathbb{J} \mathbf{w} + \int_{\Omega(t)}\rho(\mathbf{x})(\mathbf{x} - \mathbf{x}_c) \times \dot{\mathbf{v}}_D(\mathbf{x}, t) \, d\Omega \]

\[ = \int_{\Omega(t)} \rho(\mathbf{x})(\mathbf{x} - \mathbf{x}_c) \times \mathbf{f}_v(t) \, d\Omega + \int_{\Gamma(t)} (\mathbf{x} - \mathbf{x}_c) \times \mathbf{t}(\mathbf{x}, t) \, d\Gamma \]

Deformable:

\[ \int_{\Omega(t)} \rho(\mathbf{x}) \dot{\mathbf{v}}_D \cdot \hat{\mathbf{v}}_D \, d\Omega + 2\mathbf{\omega} \times \int_{\Omega(t)} \rho(\mathbf{x}) \mathbf{v}_D(\mathbf{x}, t) \cdot \hat{\mathbf{v}}_D \, d\Omega \]

\[ = -\int_{\Omega(t)} \sigma(\mathbf{x}, t) : \nabla \hat{\mathbf{v}}_D(\mathbf{x}, t) \, d\Omega(\mathbf{x}, t) + \int_{\Omega(t)} \rho(\mathbf{x}) \mathbf{f}_v(t) \cdot \hat{\mathbf{v}}_D \, d\Omega + \int_{\Gamma_f(t)} \mathbf{t}(\mathbf{x}, t) \cdot \hat{\mathbf{v}}_D \, d\Gamma \]
Floating frame

\[
\begin{align*}
&m\dot{\mathbf{v}}_c + R_c \int_{\Omega(t)} \rho(x') \dot{\mathbf{v}}'_D d\Omega = \int_{\Omega(t)} \rho(x) f_v(t) d\Omega + \int_{\Gamma(t)} t(x, t) d\Gamma \\
&\mathcal{J}' \dot{\omega}' + \omega' \times \mathcal{J}' \omega' + \int_{\Omega(t)} \rho(x') x' \times \dot{\mathbf{v}}'_D d\Omega = \int_{\Omega(t)} \rho(x') x' \times R_c^T f_v(t) d\Omega + \int_{\Gamma(t)} x' \times R_c^T t(x, t) d\Gamma \\
&\int_{\Omega(t)} \rho(x') \dot{\mathbf{v}}'_D \cdot \hat{\mathbf{v}}'_D d\Omega + 2\omega' \times \int_{\Omega(t)} \rho(x') \mathbf{v}'_D \cdot \hat{\mathbf{v}}'_D d\Omega + \int_{\Omega(t)} \sigma' : \nabla \hat{\mathbf{v}}'_D d\Omega = \int_{\Omega(t)} \rho(x) R_c^T f_v(t) \cdot \hat{\mathbf{v}}'_D d\Omega + \int_{\Gamma(t)} R_c^T t(x, t) \cdot \hat{\mathbf{v}}'_D d\Gamma
\end{align*}
\]

with \(x'\) the position, \(\omega'\) the spin vector \((\omega = R_c \omega')\) and \(\mathbf{v}'_D\) the translation velocity computed in the floating frame.
Assuming a smooth evolution of the system allows to describe the motion of each mechanical component by a semi-discretized in space system:

\[ M(q, t)\ddot{q} = F_{ext}(t) + F(q, \dot{q}, t) + R, \]

(2)

where

- \( q \in \mathbb{R}^n \) represents the vector of generalized degrees of freedom,
- \( \dot{q} \in \mathbb{R}^n \) the generalized velocities,
- \( M(q, t) : \mathbb{R}^n \mapsto \mathbb{M}^{n \times n} \) the inertia matrix,
- \( F_{ext}(t) \) the external forces,
- \( F(q, \dot{q}, t) \) the internal force (deformable bodies) and the non-linear inertia terms (centrifugal and gyroscopic),
- \( R \in \mathbb{R}^n \) the resultant of contact forces.

Initial and boundary conditions must be added to fully describe the evolution of the system.
Considering a rigid body, a more suitable dynamical equation may be used introducing $\mathbf{v}$ and $\omega$, the translation and rotation velocities of the center of mass.

Eq.(2) is replaced by the well-known Newton-Euler system of equations:

\[
\begin{align*}
M \ddot{\mathbf{v}} &= \mathbf{F}_{ext}(t) + \mathbf{R} \\
J \dot{\omega} &= -\omega \wedge (J \omega) + M_{ext}(t) + M_R
\end{align*}
\]

where

- $M$ and $J$ represents respectively the mass and the inertia matrices,
- $\mathbf{F}_{ext}(t)$ and $\mathbf{r}$ represent respectively the resultant of external and contact forces,
- $M_{ext}(t)$ and $M_R$ represent respectively the momentum due to external and contact forces.

Note that for 2D components or 3D components with geometric isotropy $\omega \wedge (I \omega)$ vanishes.
DEM : Modelling Framework : Smooth discretized dynamics

Whatever the formulation, let's consider that the dynamics of all bodies are described by:

\[ \mathbf{M} \ddot{\mathbf{V}} = \mathbf{F}_{int} + \mathbf{F}_{ext} + \mathbf{R} \]
DEM : Modelling framework : Interaction description

At any time of the evolution of the system one needs to define the interaction locus and an associated local frame in order to describe the interaction behaviour.

⇒ implicit a priori.

It is assumed that one is able to define for each point \(C\) of the candidate boundary its (unique) nearest point \(A\) on the antagonist boundary. It allows to define for each couple of points a local frame \((\mathbf{t}, \mathbf{n}, \mathbf{s})\) : with \(\mathbf{n}\) the normal vector of the antagonist boundary and \((\mathbf{s}, \mathbf{t})\) two vectors of its tangential plane.

Knowing \(C\) and \(A\) one defines:

\[
g = D(\mathbf{q})
\]

\[
\mathbf{V} = \mathbb{H}^*(\mathbf{q})\mathbf{V} = \nabla_q D(\mathbf{q})\mathbf{V}
\]

Only in simplest cases (rigid body with strictly convex boundary) the interaction locus may be considered as punctual.
More generally, using kinematic relations, one can write for a given contact $\alpha$ (between $i$ and $j$):

$$\mathbf{v}^\alpha = \mathbf{H}^{\alpha,*}(\mathbf{q}) \mathbf{v}^{ij}$$
More generally, using kinematic relations, one can write for a given contact $\alpha$ (between $i$ and $j$) :

$$\mathbf{V}^\alpha = \mathbf{H}^\alpha,\star(q)\mathbf{V}^{ij}$$

Using duality consideration (equality of power expressed in terms of global or local unknowns), the local contact force may be mapped on the global unknowns :

$$\mathbf{R}^{ij} = \mathbf{H}^\alpha(q)\mathbf{R}^\alpha,$$

where $\mathbf{H}^\alpha,\star(q)$ is the transpose of $\mathbf{H}^\alpha(q)$. 
More generally, using kinematic relations, one can write for a given contact $\alpha$ (between $i$ and $j$):

$$V^\alpha = H^{\alpha,*}(q)V^{ij}$$

Using duality consideration (equality of power expressed in terms of global or local unknowns), the local contact force may be mapped on the global unknowns:

$$R^{ij} = H^{\alpha}(q)R^\alpha,$$

where $H^{\alpha,*}(q)$ is the transpose of $H^{\alpha}(q)$.

In the following, operators mapping all the local and global unknowns are introduced:

$$H(q) : \mathcal{R} = \{R^\alpha\} \rightarrow R = \sum_{\alpha=1}^{n_c} H^{\alpha}(q)R^\alpha$$

$$H^*(q) : \mathcal{V} \rightarrow \mathcal{V} = \{V^\alpha\} = \{H^{\alpha,*}(q)V^{ij}\}$$
Assuming the occurrence of non smooth phenomena (collisions and other) the differential system must be modified.

\[ M(V^+ - V^-) = \int_{t^-}^{t^+} (F(q, V, s) + F_{\text{ext}}(s)) \, ds + I \]

where the impulse \( I \) will contain both the sum of the contribution of smooth load over the time interval \( \int_{t^-}^{t^+} R \, dt \), the percussion, denoted \( P \), at shock time (supposed instantaneous).

To describe \( V \) one uses the local bounded variation framework over all sub-intervals of \( I_T := [0, T] \), i.e. \( \text{lbv}(I_T, R) \) and one introduces differential measures allowing the generalization of the equation of motion to non smooth phenomena (Moreau 1988).
DEM : Modelling framework : Non smooth dynamics

Assuming the occurrence of non smooth phenomena (collisions and other) the differential system must be modified.

The “spirit” of the approach is to consider a weaken form of the dynamical system, e.g. balance of momentum:

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The “spirit” of the approach is to consider a weaken form of the dynamical system, e.g. balance of momentum :

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To describe $V$ one uses the local bounded variation framework over all sub-intervals of $I_{Tps} = [0, T]$, i.e. $lbv(I_{Tps}, \mathbb{R})$ and one introduces differential measures allowing the generalization of the equation of motion to non smooth phenomena (Moreau 1988).
For a given contact $\alpha$:

- **Local - global mapping**:

  \[
  \mathbf{v}^+ = H^*(q)\mathbf{v}^+
  \]
  \[
  \mathbf{l} = H(q)\mathbf{l}
  \]

- **Interaction law**:

  \[
  \text{law}(g, v^+, l) = 0
  \]

- **Shock law**:

  In the present work we consider only binary shocks using a Newton’s restitution law

  \[
  \mathbf{v}^+ N = -e_n \mathbf{v}^- N
  \]
For a given contact $\alpha$:

- Local - global mapping:
  \[
  \mathbf{v}^+ = \mathbf{H}^*(q)\mathbf{v}^+
  \]
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  \mathbf{v}^+_N = -e_n \mathbf{v}^-_N
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DEM : Numerical Strategy

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Depending on modelling choices numerical strategies are build to solve the evolution problem. They depend on:

- time evolution strategy: time stepping or event driven
- time integrator over a time step: explicit or implicit
- implicit contact solver if necessary (Lemke, Gauss-Seidel, bi-potential, etc.)
- technical aspects: contact detection, rotation integration, etc.
- etc.

Over a time step $[t, t+h]$, three important tasks can be underlined:

- The contact detection
- The computation of contact forces, called contact problem
- The motion of the different element of the media.
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Over a time step $[t, t+h]$, three important tasks can be underlined:

- The contact detection
- The computation of contact forces, called contact problem
- The motion of the different element of the media.
Time integration of the equation of motion leads to:

$$M(V_{i+1} - V_i) = \int_{t_i}^{t_{i+1}} (F(q, V, s) + F_{ext}(s))ds + I_{i+1}$$

$$= I_{free} + I_{i+1}$$

$$q_{i+1} = q_i + \int_{t_i}^{t_{i+1}} V ds$$

where

- $I_{i+1}$ represents the contact impulse over the time step
- $I_{free}$ the impulse of applied forces over the time step.
A $\theta$-method (Crank-Nicholson) is used to evaluate $\int_{t_i}^{t_{i+1}} (F(q, V, s) + F_{ext}(s)) ds$ and $\int_{t_i}^{t_{i+1}} \dot{q} ds$:

$$I_{\text{free}} = h(1 - \theta)(F(q_i, V_i, t_i) + F_{ext}(t_i)) + h\theta(F(q_{i+1}, V_{i+1}, t_{i+1}) + F_{ext}(t_{i+1}))$$

$$q_{i+1} = q_i + h((1 - \theta)V_i + \theta V_{i+1}) = q_m + h\theta V_{i+1}$$

with $q_m = q_i + h(1 - \theta)V_i$.

- If $\theta \in [0.5, 1]$ the scheme is implicit and stable unconditionally.
- If $\theta = 0.5$ the scheme is conservative for smooth evolution problem,
- If $\theta = 0$ the scheme is explicit
A $\theta$-method (Crank-Nicholson) is used to evaluate $\int_{t_i}^{t_{i+1}} (F(q, V, s) + F_{\text{ext}}(s)) ds$ and $\int_{t_i}^{t_{i+1}} \dot{q} ds$:

\[ I_{\text{free}} = h(1 - \theta)(F(q_i, V_i, t_i) + F_{\text{ext}}(t_i)) + h\theta(F(q_{i+1}, V_{i+1}, t_{i+1}) + F_{\text{ext}}(t_{i+1})) \]

\[ q_{i+1} = q_i + h((1 - \theta)V_i + \theta V_{i+1}) = q_m + h\theta V_{i+1} \]

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- If $\theta = 0$ the scheme is explicit

Time discretization is imposed arbitrarily. Its choice is mainly driven by the precision of the contact treatment.

If discontinuities occur they are treated simultaneously. No limitation on the number of interactions but the time order is lost.
The problem to solve is written in terms of:

- discretized equations of motion for each body expressed with global unknowns:
  \[ V_{i+1} = V_{\text{free}} + M^{-1}l_{i+1} \]

- interaction laws expressed with local unknowns (contact \( \alpha \)):
  \[ \text{law}(g^{\alpha}, V^{\alpha}, I^{\alpha}) = 0 \]

- mappings (\( H \) and \( H^* \)) to pass from local to global unknowns

\[ q, V \leftarrow \text{Equations of motion} \rightarrow l \]

\[ H^* \downarrow \]

\[ \uparrow H \]

\[ V \leftarrow \text{Interaction laws} \rightarrow I \]
The problem to solve is written in terms of:

- discretized equations of motion for each body expressed with global unknowns:
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- interaction laws expressed with local unknowns (contact \( \alpha \)):
  \[ \text{law}(g^\alpha, \mathbf{V}^\alpha, \mathbf{I}^\alpha) = 0 \]

- mappings (\( \mathbb{H} \) and \( \mathbb{H}^\star \)) to pass from local to global unknowns
  \[ \begin{align*}
  \mathbf{q}, \mathbf{V} & \leftarrow \text{Equations of motion} \rightarrow \mathbf{l} \\
  \mathbb{H}^\star & \downarrow \\
  \mathbb{V} & \leftarrow \text{Interaction laws} \rightarrow \mathcal{I}
  \end{align*} \]

Using basic algebraic transformations the equations of motion may be expressed in terms of local unknowns:

\[ \mathbf{V} = \mathbf{V}_{\text{free}} + \mathbf{W} \mathcal{I} \]

where \( \mathbf{W} = \mathbb{H}^\star \mathbf{M}^{-1} \mathbb{H} \) and \( \mathbf{V}_{\text{free}} = \mathbb{H}^\star (\mathbf{V}_i + \mathbf{M}^{-1} \mathbf{I}_{\text{free}}) = \mathbb{H}^\star \mathbf{V}_{\text{free}} \).
When deformable bodies are used, the local dynamic system may be written in the same way

\[ \mathbf{V} = \mathbf{V}_{\text{free}} + \mathbf{W} \mathbf{I} \]

with

- **rigid**: \( \tilde{\mathbf{M}} = \mathbf{M} \)
  \[ l_{\text{free}} = h(1 - \theta)(\mathbf{F}_i + \mathbf{F}_{\text{ext},i}) + h\theta(\mathbf{F}_f + \mathbf{F}_{\text{ext},f}) \]

- **linear**: \( \tilde{\mathbf{M}} = \mathbf{M} + h\theta \mathbf{C} + h^2\theta^2 \mathbf{K} \)
  \[ l_{\text{free}} = [\mathbf{M} - h(1 - \theta)\mathbf{C} - h^2\theta(1 - \theta)\mathbf{K}] \mathbf{u}_i - h\mathbf{K} \mathbf{q}_i + h[\theta \mathbf{F}_{\text{ext},f} + (1 - \theta)\mathbf{F}_{\text{ext},i}] \]

- **non-linear**: \( \tilde{\mathbf{M}}^k = \mathbf{M} + h\theta \mathbf{C}^k + h^2\theta^2 \mathbf{K}^k \)
  \[ l_{\text{free}} = \tilde{\mathbf{M}}^k \mathbf{u}_f^k + \mathbf{M}(\mathbf{u}_i - \mathbf{u}_f^k) + h[(1 - \theta)(\mathbf{F}_i + \mathbf{F}_{\text{ext},i}) + \theta(\mathbf{F}_f + \mathbf{F}_{\text{ext},f})] \]
DEM : Numerical Strategy : NSCD : Resolution Scheme

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Iteration matrix computation ($M$)

Halt criteria
The classical NSCD approach rely on a Non Linear Gauss Seidel (NLGS) algorithm:

\[ V_{k+1}^{\alpha} = V_{k}^{\alpha}_{\text{free}} + W_{\alpha \alpha} I_{k+1}^{\alpha} + \sum_{\beta = 1}^{\alpha-1} W_{\alpha \beta} I_{k+1}^{\alpha} + \sum_{\beta = \alpha+1}^{\text{nc}} W_{\alpha \beta} I_{k+1}^{\beta} \]

\[ \text{Law} \left( g_{k+1}^{\alpha}, V_{k+1}^{\alpha}, N_{\alpha}, T_{\alpha}, I_{k+1}^{\alpha}, N_{\alpha}, I_{k+1}^{\alpha}, T_{\alpha} \right) = 0 \]

We repeat the process until convergence.
The classical NSCD approach rely on a Non Linear Gauss Seidel (NLGS) algorithm:

Considering contacts one by one ($\alpha$), we solve the local problem:

$$\mathbf{V}^{k+1,\alpha} = \mathbf{V}^{\alpha}_{\text{free}} + \mathbf{W}^{\alpha\alpha} \mathbf{I}^{k+1,\alpha} + \sum_{\beta=1,\alpha-1}^{\alpha} \mathbf{W}^{\alpha\beta} \mathbf{I}^{k+1,\beta} + \sum_{\beta=\alpha+1,nc} \mathbf{W}^{\alpha\beta} \mathbf{I}^{k,\beta}$$

$$\text{Law}(g^{k+1,\alpha}, \mathbf{V}^{k+1}_{N}, \mathbf{V}^{k+1}_{T}, \mathbf{I}^{k+1}_{N}, \mathbf{I}^{k+1}_{T}) = 0$$
The classical NSCD approach rely on a Non Linear Gauss Seidel (NLGS) algorithm:

Considering contacts one by one ($\alpha$), we solve the local problem:

$$
\mathbf{V}^{k+1,\alpha} = \mathbf{V}_{\text{free}}^{\alpha} + \mathbf{W}^{\alpha\alpha} I^{k+1,\alpha} + \sum_{\beta=1,\alpha-1} \mathbf{W}^{\alpha\beta} I^{k+1,\beta} + \sum_{\beta=\alpha+1,\text{nc}} \mathbf{W}^{\alpha\beta} I^{k,\beta}
$$

$$
\text{Law}(g^{k+1,\alpha}, \mathbf{V}^{k+1,\alpha}_N, \mathbf{V}^{k+1,\alpha}_T, I^{k+1,\alpha}_N, I^{k+1,\alpha}_T) = 0
$$

we repeat the process until convergence
Local solvers:

2D

- explicit uncoupled resolution if $W^{\alpha\alpha}$ is diagonal
- coupled $(N, T)$ through graph intersection
- pseudo-potential approach (bi-potential)
- LCP solver
- etc.

3D

- explicit resolution if $W^{\alpha\alpha}$ is diagonal
- Generalized Newton algorithm
- pseudo-potential approach
- LCP solver
- etc
Various implementation of NLGS are possible

- Exchange Local Global (ELG); only $W$ diagonal is build
- Stored Delassus Loop strategy (SDL); $W$ is built
- A quasi NLGS may be derived to short-circuit iterative local solver
- A multi-thread Stored Delassus Loop strategy (SDL) through OpenMP
- A multi-process version through DDM and MPI

Alternatives:

- Jacobi
- Conjugate Projected Gradient Algorithm
- Lemke (for small collection of rigid bodies)
- Other possibilities, see Siconos Numerics
Convergence norm

- ad-hoc rule on the “necessary” number of iterations
- stationarity condition
  \[
  \| I_{k+1,\alpha} - I_{k,\alpha} \|_{I_{k+1,\alpha}} < \epsilon \quad \forall \alpha
  \]

- tolerance on interaction law:

  Once computed \( I_{k+1,\alpha} \), \( V_{k+1,\alpha} \) \( \forall \alpha \)

  One perform a Jacobi loop (aka solving contacts as if uncoupled): \( I_{Jac,\alpha} \), \( V_{Jac,\alpha} \) \( \forall \alpha \)

  And evaluate the distance between the two solutions.
Evolution of contact forces in the course of iterations initialized by zero force at all contacts in a system subjected to equal top and right stresses.
DEM: Interaction laws & masonry

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Frictional contact : Signorini-Coulomb

\[ I_n \geq 0 \quad g \geq 0 \quad I_n \cdot g = 0 \]
\[ \| I_t \| \leq \mu I_n, \begin{cases} \| I_t \| < \mu I_n \Rightarrow V_t = 0 \\ \| I_t \| = \mu I_n \Rightarrow \exists \alpha \geq 0, V_t = -\alpha I_t \end{cases} \]
Frictional contact: Signorini-Coulomb

\[ I_n \geq 0 \quad g \geq 0 \quad I_n \cdot g = 0 \]

\[ \| I_t \| \leq \mu I_n, \left\{ \begin{array}{ll}
\| I_t \| < \mu I_n \Rightarrow V_t = 0 \\
\| I_t \| = \mu I_n \Rightarrow \exists \alpha \geq 0, V_t = -\alpha I_t
\end{array} \right. \]

**Remark**: for dynamical problems, it is more natural to formulate the unilateral contact in terms of velocities

*Assuming* \( g(t_0) \geq 0 \) *then* \( \forall t > t_0 \)

*if* \( g(t) \leq 0 \) *then* \( V_n \geq 0, I_n \geq 0, I_n \cdot V_n = 0 \)

*else* \( I_n = 0 \)
The modelling of masonry structures with rigid blocks is particularly relevant for settlement or basic rocking.
Lets consider 2 bodies in interaction

Subscripts $N$ and $T$ denote respectively normal and tangential components of the

- stress vector $\sigma = \sigma_N n + \sigma_T$,
- cohesive stress vector $\sigma^{coh} = \sigma^{coh}_N n + \sigma^{coh}_T$
- displacement jump $[u] = u_N n + u_T$
All cohesive models coupled with non regularized contact and Coulomb’s friction may be written:

\[
\left( \sigma_N + \sigma_N^{\text{coh}} \right) \geq 0 \quad u_N \geq 0 \quad \left( \sigma_N + \sigma_N^{\text{coh}} \right) \cdot u_N = 0
\]

\[
\| \sigma_T + \sigma_T^{\text{coh}} \| \leq \mu \kappa(\beta) \left( \sigma_N + \sigma_N^{\text{coh}} \right)
\]

with:

\[
\begin{cases}
\| \sigma_T + \sigma_T^{\text{coh}} \| < \mu \kappa(\beta) \left( \sigma_N + \sigma_N^{\text{coh}} \right) \Rightarrow \dot{u}_T = 0 \\
\| \sigma_T + \sigma_T^{\text{coh}} \| = \mu \kappa(\beta) \left( \sigma_N + \sigma_N^{\text{coh}} \right) \Rightarrow \exists \alpha \geq 0, \dot{u}_T = -\alpha \left( \sigma_T + \sigma_T^{\text{coh}} \right)
\end{cases}
\]

\[
\sigma^{\text{coh}} = \Upsilon(\beta, [u]) \quad \text{and} \quad g \left( \dot{\beta}, \beta, u_N, u_T \right) = 0, \quad \beta \in [0, 1]
\]

where \( \mu \) is the Coulomb’s friction coefficient, \( \beta \) is a surface damage variable, \( \Upsilon \) is a given function associated with the shape of the cohesive zone model, \( \kappa(\beta) \) describing the evolution of friction with surface damage and \( g \) describe the evolution of damage with respect to other internal variables.
The two classes of cohesive zone models are included in this formulation:

- **intrinsic model**:
  \[
  \gamma(\beta, [u]) = K(\beta) \cdot U
  \]
  with for example
  \[
  K(\beta) = \beta \left( C_N n \otimes n + C_T \frac{u_T \otimes u_T}{\|u_T\|^2} \right),
  \]

- **extrinsic model**:
  \[
  \gamma(\beta, [u]) = \gamma(\beta)
  \]
  for instance \[\gamma(\beta) = \beta \sigma_{\text{coh max}}^\text{coh}\].

It's possible to incorporate a progressive transition from cohesion to friction giving a particular form to the function \(\kappa(\beta)\), as:

\[
\kappa(\beta) = (1 - \beta)^s \quad \text{with } s = 1, 2 \text{ or } 3, \text{ for example.}
\]
An example:

\[ \beta = \min(g(\|[u]\|), g(\|[u]\|_{\text{max}})) , \]

\[ g(x) = \begin{cases} 
\beta_0 & \text{if } x \leq \delta_0, \\
\beta_0 \frac{\delta_0}{x} \left(1 - \left(\frac{x - \delta_0}{\delta_c - \delta_0}\right)^2\right) & \text{if } \delta_0 < x < \delta_c, \\
0 & \text{if } x \geq \delta_c, 
\end{cases} \]

where \( \delta_0 = \frac{\sigma_{\text{max}}}{C} \), \( \delta_c = \frac{3}{2} \left(\frac{w}{\sigma_{\text{max}}} + \frac{\delta_0}{6}\right) \), \( 0 \leq \beta_0 \leq 1 \) is an initial surface damage, \( w \) is a reference fracture energy \( (J/m^2) \), \( \sigma_{\text{max}} \) is the maximum value of the cohesive stress \( (MPa) \), \( \|[u]\|_{\text{max}} \) is the maximum value reached by \( \|[u]\| \) during the fracture process.
In a 2D case, (a) shows respectively the normal behaviour (with $\|u_T\| = 0$) and (b) the tangential behaviour (with $\sigma_N^{coh}$ constant):

\[\sigma_N \begin{cases} 0 & \delta_c / \delta_0 - 1 \\ \sigma_{max} & \end{cases}\]

\[\|\sigma_T\| \begin{cases} 0 & \delta_c / \delta_0 - 1 \\ \|u_T\| / \delta_0 & \end{cases}\]
Several laws are already available in LMGC90:

- RCCM
- Tvergaard Hutchinson
- Monerie - Perales
- ...

Cohesive laws are mainly unidirectional and driven by scalar parameters: $\delta_0$, $\delta_c$, $w$, $\sigma_{\text{max}}$. Mixity is managed through:

- displacement jump decomposition $[u] = u_N n + u_T \Rightarrow \delta_N = <u_N>^+ ; \delta_T = ||u_T||$
- mixed initiation criteria: $\left(\frac{C_N}{\sigma_{N,\text{max}}}\delta_N\right)^2 + \left(\frac{C_T}{\sigma_{T,\text{max}}}\delta_T\right)^2 = 1 \Rightarrow \delta_0([u])$
- mixed failure criteria: $\frac{G(u_N)}{w_N} + \frac{G(u_T)}{w_T} = 1 \Rightarrow \beta; \delta_c([u])$
DEM : Conclusion

1. Context
2. Modelling Framework
3. Numerical Strategy
4. Interaction laws relevant for masonry structures
5. Conclusion
DEM : Conclusion

We have developed a simulation framework able to tackle modelling of masonry structures.

Many remaining issues:
- **technical aspects**: contact detection, domain decomposition, etc
- **modelling aspects**: material parameters, initial state, etc
- **numerical aspects**: time steps, convergence norm, indetermination

Work funded by Saladyn project (ANR-Cosinus), Degrip project (OSEO-FEDER), Geoter, Saint Gobain Research, NECS