Simple Homogenized Models for the Limit Analysis of Masonry Structures In- and Out-of-Plane Loaded

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Object of the presentation: to show simple equilibrated/kinematic models to deduce masonry failure surfaces and application at a structural level

Model 1) utilization of polynomial expansions of the micro-stress

Application to the linear elastic case and limit analysis

Model 2) limit analysis coarse FE discretization of the cell

Application to in- and out-of-plane cases

Kinematic approaches of homogenization
Theoretical/technical background:
1) Homogenization theory
2) Limit analysis (classic kinematic and static theorems) applied in combination with homogenization
3) Structural FEM analyses with rigid-plastic materials with infinite ductility

Limits of the approach:
1) Impossibility to estimate displacements at failure
2) Impossibility to evaluate structural ductility
3) Impossibility to reproduce softening behavior
4) Limitation of the model when dealing with typical non-associated phenomena (friction)

Advantages:
1) Quick estimate of failure loads and failure mechanisms
2) Avoid the utilization of heterogeneous approaches → utilization of simplified homogenization for masonry
3) Simple extension to the non-linear case (Milani 2012)
4) FEM discretization of the cell avoided
Layout of the presentation

Illustration of the first model (polynomial expansion) and application for the evaluation of in-plane strength domains

Application to structural examples

Illustration of the second model (CST discretization) and extension to out-of-plane loads

Alternative kinematic procedures
- Micro-stress equilibrium without body-forces
- Constitutive relationship for mortar and bricks (non-linear)
- First order approximation of micro-strain field (in- and out-of-plane case)
- Anti-periodicity conditions of the stress field and periodicity conditions of the periodic displacement field

\[
\begin{align*}
\text{div}\sigma & = 0 \\
\sigma & = a(y)\varepsilon \\
\varepsilon & = E + y_3\chi + \text{sym}(\text{grad}u^\text{per}) \\
\sigma e_3 & = 0 \text{ on } \partial Y_3^+ \text{ and } \partial Y_3^- \\
\sigma n & \text{ antiperiodic on } \partial Y_i \\
u^\text{per} & \text{ periodic on } \partial Y_i
\end{align*}
\]
Advantages of a simplified approach

- Inexpensive computational effort
- Possibility to handle less optimization variables $\rightarrow$ in some cases closed form solutions
- Possibility to implement a coupled meso-macro approach
- Possibility to reproduce
  - Macroscopic masonry orthotropy
  - Fragile/frictional behavior due to bed joints
- The stress assumed approach allows:
  1) To take into account a wide family of strength domains for the materials
  2) Avoid to a-priori consider a sub-class of crack patterns
Subdivision of $\frac{1}{4}$ of the elementary cell into 9 rectangular sub-domains

Polynomial stress field inside each sub-domain

$\sigma_{ij}^{(k)} = X(y)S_{ij}^T \quad y \in Y^k$

Imposition of equilibrium inside each sub domain

$\nabla \cdot \sigma = 0$

$\rightarrow$ reduction of unknowns

$\begin{cases} S_{11}^{(2k)} + 2y_1S_{11}^{(4k)} + y_2S_{11}^{(5k)} + S_{12}^{(3k)} + y_1S_{12}^{(5k)} + 2y_2S_{12}^{(6k)} = 0 \\ S_{12}^{(2k)} + 2y_1S_{12}^{(4k)} + y_2S_{12}^{(5k)} + S_{22}^{(3k)} + y_1S_{22}^{(5k)} + 2y_2S_{22}^{(6k)} = 0 \end{cases}$

Imposition of interface stress vector continuity

$\left( \hat{X}_{ij}^{(k)}(y)\hat{S}^{(k)} + \hat{X}_{ij}^{(r)}(y)\hat{S}^{(r)} \right) n_j = 0 \quad i = 1,2$

Imposition of anti-periodicity of the stress-field

Advantage: actual thickness of the joint considered

Disadvantage: need to use polynomial expressions with high degree
Model with constant stress field

P0

- Only 7 unknowns
- Applicability to the linear elastic case imposing minimum of the complementary energy

$$\nabla \Pi \xi^{(l)} = \left( \sum_{k=1}^{9} \int A^{(k)T} C^{b,m} A^{(k)} dY_k \right) \xi - \sum_{j} \int A^{(k)T} \bar{u} dS_j = 0$$

- Advantage: good results in the linear case
- Disadvantage: orthotropy at failure lost

$$C_{\text{hom}} \xi = GE$$

$$S = \frac{1}{G_b} (e_y a + b a) + \frac{1}{G_m} \left( e_y a + e_y e_h + \frac{1}{2} e_h b \right)$$
Results in the linear elastic range (1/3)

- Results obtained on standard Italian bricks and 10 mm thick mortar joints
- Minimization of the complementary energy varying mortar Young modulus keeping the brick modulus fixed

\[ \nabla \Pi \psi^{(i)} = \left( \sum_{k=1}^{9} \int \boldsymbol{A}^{(k), T} \mathbf{C}_{b,m} \boldsymbol{A}^{(k)} dY_{k} \right) \xi - \sum_{j} \int \boldsymbol{A}^{(k), T} \mathbf{u} dS_{j} = 0 \]

A1111: horizontal membrane stiffness  
A1122: out-of-diagonal term
Results in the linear elastic range (2/3)

- Comparison with FEM (Anthoine 1995) assumed as reference solution
- A2222: always rigorously correct
- A1111: bricks staggering increases horizontal stiffness → need to use more variables

A1212: shear modulus

A2222: vertical membrane stiffness
Results in the linear elastic range (3/3)

- Shear stress component provided by the models with increasing degree of the polynomial expansion (Eb/Em=90, ¼ elementary cell)
- Need to reproduce the shear action into bed joint in horizontal stretching (Zucchini & Lourenço 2002)

Important (not all) requirements: infinite ductility of the materials, associated flow rule and rigid-perfectly plastic behavior of the material.

Advantages:
- Limit analysis approach requires only a reduced number of material parameters.
- A model with few variables allows to reproduce 3D failure surfaces very quickly.
- Classic Limit analysis problem may be handled numerically by means of consolidated linear programming codes.
- Materials are not infinite ductile and typically masonry in shear behaves with non-associated flow rules in the mortar joint.
Limit analysis: static approach

- Application to a wide variety of materials, including reinforced fibers (Taliercio and de Buhan) and masonry
- Shom: homogenized failure surface
- Imposition of micro-equilibrium in continuum
- Imposition of micro-equilibrium in internal interfaces
- Imposition of anti-periodicity on cell boundaries of the micro-stress
- Admissibility conditions

To find a point on the failure surface it is necessary to maximize the micro-stress field within the whole equilibrated and admissible fields.

\[
S_{\text{hom}} = \left\{ \sum \left\{ \begin{array}{l}
\sum = \langle \sigma \rangle = \frac{1}{A_y} \int \sigma dY \\
\text{div} \sigma = 0 \\
[\sigma] h^{\text{int}} = 0 \\
\sigma_n \text{ anti-periodic on } \partial Y \\
\sigma(y) \in S^m \quad \forall y \in Y^m ; \quad \sigma(y) \in S^b \quad \forall y \in Y^b
\end{array} \right\} \right\}
\]
Limit analysis: kinematic approach

- Requires very few variables in case of infinitely resistant bricks and joints reduced to interfaces
- Imposition of admissibility conditions for the plastic flow: associated flow rule in continuum
- Imposition of periodicity of the velocity field
- To find a point on the homogenized failure surface it is necessary to minimize the internal power dissipation within the whole class of the possible failure mechanisms

\[
S_{\text{hom}} = \sum \left\{ \begin{aligned}
\sum : \mathbf{D} &\leq \pi_{\text{hom}}(\mathbf{D}) \quad \forall \mathbf{D} \in \mathbb{R}^6 \\
\pi_{\text{hom}}(\mathbf{D}) &= \inf_{\mathbf{P}(\mathbf{v})} \left\{ \mathbf{P}(\mathbf{v}) | \mathbf{D} = \frac{1}{2} \int_{\partial^Y} (\mathbf{v} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{v}) dS \right\} \\
\mathbf{P}(\mathbf{v}) &= \int_{Y} \pi(\mathbf{d}) dY + \int_{S} \pi([\mathbf{v}] ; \mathbf{n}) dS
\end{aligned} \right\}
\]
Limit analysis: static approach evaluation of one point on the homogenized failure surface

- Choice of a direction in the homogenized stress space $n_\Sigma = [\alpha_{11} \quad \alpha_{22} \quad \alpha_{12}]^T$
- Optimization variable: distance of the failure surface from the origin along the chosen direction
- Typical non-linear programming problem where the non-linearity is due to material (brick and mortar) strength domains
- If the brick is infinitely resistant and joint is reduced to interface with Mohr-Coulomb failure criterion – eventually with tension cutoff – the problem becomes linear

\[
\begin{align*}
\max \{ \lambda \} \quad \text{such that} \\
\lambda n_\Sigma = \lambda \begin{bmatrix} \alpha_{11} & \alpha_{22} & \alpha_{12} \end{bmatrix}^T = \frac{1}{Y} \int_{\partial Y} \sigma dY \\
\sigma \text{ anti-periodic on } \partial Y \\
\text{div} \sigma = 0 \\
\sigma(y) \in S'(y) = \begin{cases} 
S^b & \text{if } y \in \text{blocks} \\
S^m & \text{if } y \in \text{mortar}
\end{cases}
\end{align*}
\]
Approximate imposition of the admissibility condition in case of polynomial expansion

- Admissibility imposed in a regular grid of points $\rightarrow$ quasi lower bound solution
- Experienced very fast convergence $\rightarrow$ only few points are necessary
- As expected, convergence always found with a decrease of the multiplier refining grid

\[
\max\{\lambda\} \quad \text{such that} \quad \lambda n_\Sigma = \frac{1}{Y} \sum_i \int_Y \tilde{X}^{(k)}(y)\tilde{S}dY
\]

\[
\tilde{S} = \tilde{X}^{(k)}(y)\tilde{S}
\]

\[
\tilde{\sigma}^j \in S^i \quad j = 1, ..., rq
\]

\[
i = 1, ..., 4k^{\max}
\]

\[
\text{nodal points}
\]

\[
\lambda \text{ multiplier}
\]

Horizontal subdivision (square grid)
Utilization of simplex method (simplex find always the optimal solution on one corner of the failure surface)

Coarse upper-bound approximation of the non-linear convex strength domain

Refinement of the failure surface at the successive iteration only where needed

Convergence to the actual final solution very quickly
Homogenized failure surfaces: tension-tension regime

- Results for common Italian bricks in the tension tension-region
- Comparison with FEM elasto-plastic
- Bricks infinitely resistant
- Mortar joints with a plane-stress Mohr-Coulomb failure criterion
- Model P0 exhibiting no orthotropy → unsuitable for limit analysis computations
- Orthotropy due to bed joint working in shear
Results for common Italian bricks in the tension tension-region
Comparison with de Buhan & de Felice approach
Bricks infinitely resistant
Mortar joints reduced to interfaces with pure MC failure criterion
Results at different orientations of the bed joints with respect to one of the principal homogenized stresses (horizontal component)
Three orientations inspected: 0°, 22.5°, 45°
Homogenized failure surfaces: comparison with a kinematic approach in compression

- Utilization of a linearized Lourenço-Rots (1997) failure criterion for joints reduced to interfaces
- 5 parameters strength model (friction angle, cohesion, tensile strength, compression strength, shape of the linearized compressive cap)
- 3 models compared
  - Model A: vertical compression cutoff
  - Model B: elliptic compressive cap
  - Model C: joints in plane stress condition

<table>
<thead>
<tr>
<th>Model A</th>
<th>Model B</th>
<th>Model C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_i$ (c=1.4 $f_i$)</td>
<td>$f_c$</td>
<td>$f_i$ (c=1.4 $f_i$)</td>
</tr>
<tr>
<td>0.16 [N/mm²]</td>
<td>11.5 [N/mm²]</td>
<td>0.16 [N/mm²]</td>
</tr>
<tr>
<td>$\Phi_1$</td>
<td>$\Phi_2$</td>
<td>$\Phi_1$</td>
</tr>
<tr>
<td>37°</td>
<td>90°</td>
<td>37°</td>
</tr>
</tbody>
</table>

$$c = \sigma_c \left( \frac{1 - \sin(\Phi)}{2\cos(\Phi)} \right)$$

$$\sigma_t = \frac{2c \cos(\Phi)}{1 + \sin(\Phi)}$$
Comparison with a kinematic approach with both manual solution and linear programming approach

Optimization variables: homogenized stress (with stress direction a-priori chosen) and jump of velocities on interfaces

Only equalities optimization problems

\[
\chi = \min_{\mathbf{v}} \frac{1}{\Gamma} \int_{\Gamma} [[\mathbf{v}]] \mathbf{\sigma} ds
\]

\[
\sum^0 : \mathbf{D} = 1
\]

\[
[[\mathbf{v}]] = \sum_{i=1}^{n} \hat{\mathbf{v}}_i \nabla_i \mathbf{v}^{(i)}
\]
Model B results: comparison with a kinematic approach

- Effect of the compressive cap on the shape of the compression regime failure surface → biaxial stress state
- Uniaxial strength is obviously the same
- Very little orthotropy in uniaxial vertical/horizontal compression → low cohesion when compared with compressive strength
Homogenized failure surfaces: comparison with experimental data

- Experimental data by Page: small but perceivable orthotropy in compression (10/8) → high cohesion of the mortar joint
- Limitation of the model: impossibility to reproduce the typical masonry behavior in compression (compression strength is a non-linear combination of brick and mortar compression strength)
Role played by the simplification assumed for joints→ reduction to interfaces

- Masonry strength domain is constituted by planes, whereas when the actual thickness of the joints is modeled (10 mm in the figure) the resultant failure surface is smooth and non-linear.
The macroscopic tensor of strain rate is given with the incremental homogenization problem

\[
\begin{align*}
\dot{\sigma}_{ij,j} &= 0 \\
\dot{u}_i &= \dot{E}_{ij} y_j + \dot{u}_{i}^{\text{per}} \quad \text{on } Y \\
\dot{\sigma}_{ij} &= \begin{cases} 
\Psi^b(\varepsilon) \\
\Psi^m(\varepsilon) 
\end{cases} \\
\dot{\sigma}_{ij} &\text{ periodic on } Y
\end{align*}
\]

The incremental problem is solved by means of the constrained minimization problem of the Capurso-Maier’s (1970) functional

\[
\Pi(\dot{\sigma}_y, \dot{\lambda}) = \frac{1}{2} \int_Y C_{\dot{y}h} \dot{\sigma}_y \dot{\sigma}_{hk} dY + \frac{1}{2} \int_Y \dot{\lambda} \frac{\partial g^{h,m}}{\partial \sigma_{y}} \dot{\sigma}_{y} dY - \int_{\partial Y} \dot{\sigma}_{I} n_i \dot{u}_j dS
\]

→ simple quadratic programming problem on the strain rate increment
For elastic-perfectly plastic materials the failure surface is evaluated point by point varying the strain rate ratio between vertical and horizontal component → homogenized stress rate is not constant

- as the “final” homogenized stress state of the simulations, i.e. when a failure mechanism is reached
- Simulations with Mohr-Coulomb failure criterion for mortar and Rankine failure criterion for bricks
- It is also visible the difference between the elastic-limit surface and the strength domain
- → ductility of the structure
Generalization to shear strain rate

- Drastic increase of the time needed for the simulations
- Evaluation of the failure surface through tessellation
- Comparison with de Felice and de Buhan (1997) approach
Structural applications: lower bound FEM limit analysis

- Triangular equilibrated element by Sloan (1988)
- Linear interpolation of the stress tensor inside an element
- Continuity of the stress vector on interfaces
- Element used for the limit analysis of r.c. slabs (Poulsen & Damkilde 2002), steel (Olsen 1999, Olsen 2001) and masonry (Sutcliffe et al. 2002)

\[
\mathbf{H} = \begin{bmatrix}
 h_1 & 0 & 0 \\
 0 & h_2 & 0 \\
 0 & 0 & h_3 \\
 h_1^M & h_2^M & h_3^M
\end{bmatrix}
\]

\[
\mathbf{q}_r = \begin{bmatrix}
 \frac{(y_k - y_r)^2}{l_{kr}} \\
 -\frac{(x_k - x_r)(y_k - y_r)}{l_{kr}} \\
 \frac{(y_k - y_r)^2}{l_{kr}} \\
 -\frac{(x_k - x_r)(y_k - y_r)}{l_{kr}} \\
 \frac{(y_r - y_j)^2}{l_{rj}} \\
 -\frac{(x_r - x_j)(y_r - y_j)}{l_{rj}} \\
 \frac{(y_r - y_j)^2}{l_{rj}} \\
 -\frac{(x_r - x_j)(y_r - y_j)}{l_{rj}}
\end{bmatrix}
\]

\[
\Sigma^{(r)} = \mathbf{h}_r \Sigma^{(r)}
\]
Structural level: homogenized limit analysis

- Triangular element by Sloan & Kleeman (1995)
- Dissipation allowed in continuum and on interfaces
- Linear interpolation of the velocity field inside each element
- Linear interpolation of the velocity field on interfaces
- Optimization variables: elements velocities, elements and interfaces plastic multipliers
- Direct output of the code: field of velocities at collapse, upper bound estimate of failure load
- Indirect output of the code: stress distribution from the solution of the dual problem
LP numerical algorithms for limit analysis

- Simplex method (Revised Simplex Method) → Dantzig (1947, 1963) → exponential complexity
- Interior point method
  - “Ellipsoid Method” by Khachiyan (1979) → polynomial complexity (O(n^2L)) → in practice not suitable
  - Karmarkar (1984) algorithm → polynomial complexity (O(nL)) → very good performance in case of problems with many variables and sparse matrices
  - “Predictor-Corrector” algorithms → very diffused in commercial codes → Mehrotra (1992) algorithm
- Non linear programming based on conic programming (Sloan 2008, Krabbenhoft 2011, etc.)
• Comparison with heterogeneous approaches (Sutcliffe, Yu, Page 2001 CAS)

• Formation of inclined compressed struts

<table>
<thead>
<tr>
<th></th>
<th>Collapse Load [kN]</th>
<th>Number of elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Homogenous approach</td>
<td>93.5</td>
<td>288 (mesh 24x6)</td>
</tr>
<tr>
<td>Heterogeneous approach</td>
<td>105.2</td>
<td>708</td>
</tr>
<tr>
<td>Experimental</td>
<td>109.2</td>
<td>-</td>
</tr>
</tbody>
</table>
Deep beam test by Page (1981): upper bound solution

- Comparison with heterogeneous approaches (Sutcliffe, Yu, Page 2001 CAS)
- Local shear failure of the bricks under the load, crushing near the supports

### Homogenization

### Heterogeneous with discontinuities on joints and bricks

<table>
<thead>
<tr>
<th></th>
<th>Collapse Load [kN]</th>
<th>Number of elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Homogeneous approach</td>
<td>115.9</td>
<td>400 (mesh 20x10)</td>
</tr>
<tr>
<td>Heterogeneous approach</td>
<td></td>
<td></td>
</tr>
<tr>
<td>with discontinuities on</td>
<td>117</td>
<td>1056</td>
</tr>
<tr>
<td>units and joints</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Heterogeneous approach</td>
<td></td>
<td></td>
</tr>
<tr>
<td>without discontinuities</td>
<td>148</td>
<td>528</td>
</tr>
<tr>
<td>on units</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Convergence study

Collapse Load (kN)

- u.b. continuous
- u.b. with discontinuities
- lower bound

x per y subdivision

2x4 4x8 8x16 16x32

60 80 100 120 140 160 180 200 220 240
Shear walls

<table>
<thead>
<tr>
<th>Vertical load p [N/mm²]</th>
<th>Homogeneous approach</th>
<th>Heterogenous approach</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.30</td>
<td>24.22</td>
<td>26.5</td>
<td>50</td>
</tr>
<tr>
<td>1.21</td>
<td>54.63</td>
<td>57.5</td>
<td>72</td>
</tr>
<tr>
<td>2.12</td>
<td>80.81</td>
<td>82.9</td>
<td>97</td>
</tr>
</tbody>
</table>

P1=0.3MPa  P2=1.21MPa  P3=2.12MPa

Application to shear walls, where vertical pre-compression is variable
Role played by the vertical load in the dissipation due to dead loads (subtracting the internal dissipation in the objective function)
Analysis of a five story wall horizontally loaded

- Application to a large scale structure
- Immediate evaluation of failure mechanisms and collapse loads

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<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Lower Bound</td>
<td>691 kN</td>
</tr>
<tr>
<td>Upper Bound</td>
<td>787 kN</td>
</tr>
<tr>
<td>Brencich et al.</td>
<td>750 kN</td>
</tr>
</tbody>
</table>

Lower bound approach

<table>
<thead>
<tr>
<th>A</th>
<th>A</th>
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<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

thickness

220 mm

240 mm

240 mm
Model 2: discretization with CST elements

- Subdivision into 6x4=24 CST constant stress triangles
- Reduction of joints to interfaces
- Need of reproduction of shear action acting on the bed joint in uniaxial horizontal tension

![Diagram showing discretization and stress acting on elements and equilibrium on interfaces]
Formulation of the LP problem

- Unknowns for ¼ cell: 3x6=18 stress components and 1 failure load
- Equilibrium equations: CST already in equilibrium + 5 interfaces $\rightarrow$ 10 equilibrium equations on interfaces

\[
\sigma_{xx}^{(3)} = \sigma_{xx}^{(2)} - \rho (\tau^{(2)} - \tau^{(3)}) \\
\sigma_{yy}^{(3)} = \sigma_{yy}^{(2)} - \rho^{-1} (\tau^{(3)} - \tau^{(2)}) \\
\sigma_{xx}^{(2)} = \sigma_{xx}^{(1)} - \rho (\tau^{(1)} - \tau^{(2)}) \\
\sigma_{yy}^{(3)} = \sigma_{yy}^{(1)} - \rho^{-1} (\tau^{(1)} - \tau^{(2)}) \\
\sigma_{xx}^{(3)} = \sigma_{xx}^{(2)} + \rho (\tau^{(2)} - \tau^{(3)}) \\
\sigma_{xx}^{(2)} = \sigma_{xx}^{(1)} + \rho (\tau^{(1)} - \tau^{(2)}) \\
\sigma_{yy}^{(3)} = \sigma_{yy}^{(1)} + \rho^{-1} (\tau^{(1)} - \tau^{(2)}) \\
\sigma_{yy}^{(2)} = \sigma_{yy}^{(1)} + \rho^{-1} (\tau^{(1)} - \tau^{(2)}) \\
\tau^{(2)} = \tau^{(2)}
\]

- Anti-periodicity conditions: 8 equations $\tau^{(1)} = \tau^{(3)} = \tau^{(1')} = \tau^{(3')} = 0$
- $\rightarrow$ only 1 unknown $\rightarrow$ graphic solution of the linear programming problem

\[
\Sigma_{xx} = \frac{\sigma_{xx}^{(1)} + \sigma_{xx}^{(1')}}{2} \\
\Sigma_{yy} = \frac{\sigma_{yy}^{(1)} + \sigma_{yy}^{(1')}}{2} = 0
\]
Uniaxial horizontal tension

- Comparison with a kinematic approach with both manual solution and linear programming approach
- Optimization variables: homogenized stress (with stress direction a-priori chosen) and jump of velocities on interfaces
- Only inequalities optimization problems

\[
\begin{align*}
\text{max} & \left\{ \frac{\sigma_{xx}^{(1)} + \sigma_{xx}^{(1')}}{2} \right\} \\
\text{subject to} & \begin{cases} 
\tau^{(2)} \leq c & \text{(I)} \\
\sigma_{xx}^{(1)} - 2\rho \tau^{(2)} \leq f_t & \text{(II)} \\
\sigma_{xx}^{(1')} \leq f_t & \text{(III)}
\end{cases}
\end{align*}
\]
Biaxial stress state

- Comparison with a kinematic approach with both manual solution and linear programming approach
- Optimization variables: homogenized stress (with stress direction a-priori chosen) and jump of velocities on interfaces
- Only equalities optimization problems

\[
\max \{ \lambda \} \quad \text{subject to} \quad \left\{ \begin{array}{l}
\tau^{(2)} + \lambda \tan \Phi \sin \alpha \leq c \\
- \tau^{(2)} + \lambda \tan \Phi \sin \alpha \leq c \\
\lambda \sin \alpha \leq f_i \\
\sigma_{xx}^{(1)} - 2 \rho \tau^{(2)} \leq f_i \\
\sigma_{xx}^{(1)} \leq f_i
\end{array} \right. 
\]

Admissible domain
Optimal point
Homogenized failure surface
In the general case of shear actions acting → 6x4 = 24 CST elements → 24x3 = 72 stress variables + collapse multiplier

Optimization variables: 73

3 homogenization equations

27 interfaces → 54 equilibrium equations

14 anti-periodicity conditions

→ 5 optimization variables

\begin{align*}
\max \lambda \\
\lambda \alpha &= \frac{\sum_{i=1}^{24} \sigma_{xx}^{(i)} A_i}{2ab} \\
\lambda \beta &= \frac{\sum_{i=1}^{24} \sigma_{yy}^{(i)} A_i}{2ab} \\
\lambda \gamma &= \frac{\sum_{i=1}^{24} \tau^{(i)} A_i}{2ab} \\
\end{align*}

subject to

\begin{align*}
A_{eq}^l X &= b_{eq}^l \\
A_{eq}^{op} X &= b_{eq}^{op} \\
f_E^{l} \left( \sigma_{xx}^{(i)}, \sigma_{yy}^{(i)}, \tau^{(i)} \right) &\leq 0 \ \forall i = 1, ..., 24 \\
f_I^{l} \left( \sigma_I^{(i)}, \tau_I^{(i)} \right) &\leq 0 \ \forall i = 1, ..., 32
\end{align*}
Same problem analyzed with Model 1
In this case, the superposition between lower and upper bound is perfect → same results
By means of the theorems of limit analysis it can be stated that the solution found is the actual one
Model 2: extension to out-of-plane loads

- Subdivision along the thickness into layers.
  - For each layer 72 unknowns
  - → fast convergence with few layers (10)
- Less than 1000 optimization variables
  - → efficient solutions with the interior point algorithm

\[ \max \{ \lambda \} \]

\[
\begin{align*}
\lambda \alpha &= \frac{\sum_{j=1}^{n_2} \Delta L \left( \frac{t + \Delta L}{2} - j \Delta L \right) \sum_{i=1}^{24} \sigma_{xx}^{(i,j)} A_i}{2ab} \\
\lambda \beta &= \frac{\sum_{j=1}^{n_2} \Delta L \left( \frac{t + \Delta L}{2} - j \Delta L \right) \sum_{i=1}^{24} \sigma_{yy}^{(i,j)} A_i}{2ab} \\
\lambda \gamma &= \frac{\sum_{j=1}^{n_2} \Delta L \left( \frac{t + \Delta L}{2} - j \Delta L \right) \sum_{i=1}^{24} \tau_{x,y}^{(i,j)} A_i}{2ab}
\end{align*}
\]

subject to:

\[
\begin{align*}
\mathbf{A}^i \mathbf{X} &= \mathbf{b}^i \\
\mathbf{A}^{ap} \mathbf{X} &= \mathbf{b}^{ap} \\
f^j_{k}(\sigma_{xx}^{(i,j)}, \sigma_{yy}^{(i,j)}, \tau_{x,y}^{(i,j)}) &\leq 0 \\
f^j_{l}(\sigma_{x}^{(i,j)}, \tau_{x}^{(i,j)}) &\leq 0
\end{align*}
\]

\[ \forall i = 1, ..., 24 \quad \forall j = 1, ..., n_L \]

\[ \forall i = 1, ..., 32 \quad \forall j = 1, ..., n_L \]
Comparison with a kinematic approach with both manual solution and linear programming approach (Cecchi, Milani & Tralli 2007)
Comparison with a MC failure criterion for joints
Possibility to consider any failure criterion for joints
Failure surfaces represented in the $M_{xx}$-$M_{yy}$ and $M_{xx}$-$M_{xy}$ planes
Role played by vertical pre-compression

In case of limited compressive strength of the mortar joint, ultimate bending firstly increases and then decreases near the value of ultimate compression reached by the intrados layers.
Structural application

- Walls experimentally tested in two-way bending.
- Comparison with a heterogeneous approach (Milani 2008 IJ MS) and with nonlinear procedures.
Extension to full 3D analyses (entire buildings)

- Number of optimization variables incremented
- Simplified coupling between in- and out-of-plane loads
- Utilization of triangular elements with discontinuities on interfaces for in-plane loads
- Utilization of KL plate elements for out-of-plane loads
The kinematic model

- Each brick interacts with its 6 neighbors through interfacial actions.
- Bricks are supposed infinitely resistant and joints are reduced to interfaces.
- Each brick is identified in its movement by a velocity vector and a rotation matrix:
  \[ w^a(x) = w^a(p) + W^a(\xi - p) \]
  \[ w^b(x) = w^b(p) + W^b(\xi - p) \]
- First order identification between classes of regular motions in the discrete system and motion in the continuous Reissner-Mindlin plate:
  \[ w^b(x) = w(x) + \text{grad } w(x)(g^b - x) \]
  \[ W^b(x) = W(x) + \text{grad } W(x)(g^b - x) \]
Class of regular motions investigated

- Horizontal bending (involved both head and bed joints)
- Vertical bending (involved only bed joints)
- Torsional moment (head and bed joints)
- Out-of-plane 13 shear (involved both head and bed joints)
- Out-of-plane 23 shear (only bed joints)

Internal power dissipated is easily evaluated at the interfaces as a function of macroscopic curvatures and shear deformations.

\[
P_{\text{int}}' = \int \left[ v \right]^T \sigma dA' = \int_{\Omega'} \sum_{i=1}^{n_i} \hat{\lambda}_i' \left( \xi_1, \xi_2 \right) \left[ \frac{\partial \phi}{\partial \sigma} \right]^T \sigma dA' = \frac{1}{3} \sum_{i=1}^{n_i} c_i' \sum_{k=1}^{3} \hat{\lambda}_i' \left( \xi_1^R, \xi_2^R \right) A_i'
\]
Limit analysis approach: the non-linear optimization problem at a cell level

\[
\begin{align*}
\lambda &= \min_{x=(\mathbf{D},\lambda^I(P_k))} \sum_{i=1}^{n_{\text{int}}} P^I_{\text{int}} - \Sigma_0^T \mathbf{D} \\
\Sigma_1^T \mathbf{D} &= 1 \\
M'(P_k) \mathbf{D} &= \left[v^I(P_k)\right] = \sum_{i=1}^{n_{\text{fin}}} \lambda^I_i \left(\xi_{2}\xi_{P_k}, \xi_{2}\xi_{P_k}\right) \frac{\partial \phi}{\partial \sigma}
\end{align*}
\]

- For joints a general non linear failure surface can be used
- Variables are represented by homogenized internal actions (moments, out-of-plane shears and membrane actions) and plastic multipliers of the interfaces
- Simple linear (or iteratively linear, see Milani Lourenço Tralli 2006) in few variables • simplex type or active set algorithms can be used
Homogenized failure surfaces: role played by vertical pre-compression

- Comparison with a kinematic approach when $M_{xy}=0$ [$c=1 \text{ N/mm}^2$, $F=36^\circ$] - Sab 2003
- Comparisons with incremental FE procedures - Lourenço 1997 1999
- Comparison with experimental data at different load orientations $q$ with respect to the bed joint - Gazzola et al. and Gazzola Drysdale 1986, Van Der Pluijm 1995
Homogenized failure surfaces: influence of the out of plane shear

Lourenço Rots [1997] failure criterion

- M11-M22 macroscopic masonry polytope sections at assigned membrane compressive load and varying out of plane shear T13
- Sections strongly different between the two models
- Possibility to reproduce Johansen [1963] out-of-plane failure criterion for concrete
- Possibility to reproduce Kirchhoff-Love plates as a particular case of the model proposed

Mohr-Coulomb failure criterion
Three models for the evaluation of masonry strength domains in various conditions (in- and out-of-plane loads)

Procedure robust with a direct implementation on FEM structural limit analysis codes

Procedure extended to vaults, entire building and strengthened panels

Procedure extended to non-linear materials
Thank you for your kind attention

Questions?

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