Some physical limitations in the performance of statistical multiple-input multiple-output RADARs

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Abstract: The objective of the study is to clarify some physical limitations in the performance of statistical RADAR systems using multiple antennas. In particular, it is shown that the performance of the recently proposed statistical multiple-input multiple-output (MIMO) RADAR is limited by the Number of Degrees of Freedom (NDF) of the scattered field. Furthermore, the true nature of statistical RADAR systems (single-input multiple-output, multiple-input single-output, MIMO) is not related to the number of transmitting/receiving antennas, but to the NDF of the scattered field exploited by the system.

1 Introduction

Over the last 10 years new ideas and approaches have been dramatically changing the world of wireless communications, breaking or overturning a number of old paradigms. This 'new age' of communication has its roots in some recent results obtained in the field of information theory, and in particular in the introduction of spatial information besides the temporal information [1].

Very recently, the advantages of the use of spatial information have been clearly shown also in the field of RADAR systems, with the introduction of the statistical multiple-input multiple-output (MIMO) RADAR (in the following denoted as SMR) [2, 3].

It is known that slow fluctuations of the radar cross section (RCS) target (e.g. the ‘Swerling case 1’ [4]) degrade the RADAR performance more than fast fluctuations (e.g. the ‘Swerling case 2’ [4]). This is due to the fact that in the Swerling case 2, the received signal is decorrelated from pulse to pulse, whereas in the Swerling case 1, the signal is decorrelated from scan to scan, but highly correlated from pulse to pulse. The idea at the basis of the statistical MIMO RADAR is to take advantage of the ‘spatial scintillation’ of the RADAR target to improve the RADAR performance. In order to reach this goal, in MIMO RADAR, a number of transmitting (TX) and receiving (RX) antennas are distributed over a large area allowing to observe the RCS target from different angles, potentially obtaining decorrelated signals. This is equivalent to transform a Swerling case 1 RCS target to a Swerling case 2 by a proper processing of the space–time signals.

The understanding of the physical limitations regarding the gain achievable by statistical MIMO RADARs is of great interest. However, the spatial model usually adopted is quite simple and consists in a pure statistical approach in which the input–output relationship between the TX and RX antennas is modelled by a random matrix [2, 3]. This approach makes the problem mathematically tractable, but loses the physics of the problem.

The aim of this paper is to introduce a different approach allowing to identify some physical limitations regarding SMR.

As a preliminary step, it is useful to point out some strong parallelisms between statistical MIMO RADAR and multiple antennas communication systems.

With reference to communication systems, multiple antennas have been usually used to increase the signal/noise ratio (SNR), obtaining the so-called ‘array gain’ [5]. This approach requires highly correlated signals on the antennas.
However, in the presence of deep fading, there is a significant probability that the signal is weak, and this affects the reliability of the communication. A different approach can be followed provided that the signals on the antennas are highly decorrelated. In fact, in this case, there is a high probability that not all the signals are buried into the noise in the presence of fading, and hence it is possible to improve the reliability of the communication, obtaining the so-called ‘diversity gain’ [5]. Even if diversity gain can be obtained also if there are multiple antennas only at the transmission side (multiple-input single-output systems or MISO systems), or only at reception (single-input multiple-output systems or SIMO systems), the highest diversity gain is obtained using multiple antennas at both transmission and reception (MIMO systems).

In the classic approach to MIMO communication, the input–output relationship between the TX and RX antennas is modelled by means of a random matrix [5]. This pure mathematical approach gives some unphysical results, for example, in the case of densely packed antennas [6].

Recently, a novel approach has been proposed [6, 7] to identify the physical limitations in MIMO communication systems, based on the concept of the number of degrees of freedom (NDF) of the electromagnetic field. Broadly speaking, the NDF of the field is the minimum number of linearly independent functions required to reconstruct the field within a desired accuracy.

The concept of NDF is widely used in information theory to evaluate the capacity of continuous temporal channels, following the approach shown by Shannon in his famous work [8]. However, Gabor [9] discussed the limitations of the communication systems introducing a concept analogue to the NDF almost two decades before Shannon.

The concept of NDF was introduced in optics by Toraldo di Francia in 1955 [10], attracting the interest of prominent researchers, like Ronchi, Gori, Wolf, Devaney, Miller and many others, who gave important contributions to characterise optical systems from an informational point of view. The concept of NDF was introduced in the antennas and propagation community by Bucci and Franceschetti in [11] and successively applied to many different problems, including antenna characterisation and microwave tomography. Very recently, the NDF of the field was used to identify the physical limitations in the amount of information conveyed by space–time communication systems [6, 12], allowing to ‘blend’ information theory and electromagnetic theory [13].

Coming back to multiple antenna RADARs (MA-RADARs) and paralleling the discussion regarding MIMO communication systems, it is noted that classic RADARs based on multiple antennas use highly correlated received signals to improve the SNR, obtaining an array gain, whereas statistical MIMO RADARs use decorrelated signals to improve the average SNR in the presence of slow fading, obtaining a diversity gain [3].

These similarities suggest to extend the approach developed for multiple antenna communication systems in [6] to statistical MA-RADARs, with particular reference to statistical MIMO RADARs.

Some preliminary results of this research were discussed in [14].

In order to reach the above-mentioned goal, it is necessary to develop a suitable spatial model of the RADAR signal propagation. This is the aim of Section 2. In Section 3, the role of the NDF of the scattered field [11, 15, 16] in MA-RADAR, including MIMO RADAR, is discussed.

Section 4 introduces a bandlimited approximation of the scattered field based on the results reported in [17, 18] suitable for a simple and effective evaluation of the spatial autocorrelation function of the scattered field.

In Section 5, the role of the NDF of the field in obtaining a reliable SNR in the Swerling case 1 is discussed, showing that the number of spatially low-correlated signals, and consequently, the performance of the statistical RADAR system is limited by the NDF of the field falling in the area covered by the RADAR antennas. In practice, a statistical MA-RADAR that exploits only one degree of freedom works like a SISO RADAR, in spite of the use of multiple antennas, since the signals received by the antennas are highly correlated giving only an array gain. Some numerical examples are reported. Finally, conclusions are drawn in Section 6.

It is worth stressing that the results outlined in this paper are not only interesting from a theoretical point of view, but have relevant practical consequences. In fact, the data processing in the case of highly correlated received signals is different from the elaboration required in the case of highly decorrelated signals.

In this paper, a simple 2D model is discussed. However, the approach can be easily extended to a 3D geometry, since the theories regarding the NDF and bandlimited approximation of the field are valid also in the full 3D vector case [11, 17, 18].

### 2 Electromagnetic model of MA-RADAR signal propagation

The aim of this section is to introduce a model of the MA-RADAR focused on the electromagnetic propagation, extending the model discussed in [6] to RADAR applications.
Consider the MA-RADAR 2D model shown in Fig. 1, consisting of an arbitrarily large number of TX/RX RADAR antennas placed in a domain $F$ limited by a curve $\Omega$ and an ensemble of targets placed in a domain $D$ limited by a curve $\Sigma$. For the sake of simplicity, scalar fields are considered.

In the following, the signal is supposed to be narrowband. Under this hypothesis, the spatial information is related only to the carrier frequency, and it is possible to undertake an analysis using harmonic signals at the carrier frequency. In the case of wideband signals, as often happens in modern RADAR systems, it is possible to divide the frequency range to obtain a number of narrowband signals and to apply the following analysis to each of them.

Suppose a continuous set of antennas. In the following discussion, one can operate in a $L_2$ Hilbert space equipped with the usual norm.

With reference to Fig. 2, the incident field on the target $E_i$ is related to the current density $J_p$ of the sources placed in $F$ by the integral operator indicated as $A_T$ in Fig. 2, whose explicit expression is

$$E_i(r) = \int_F G(r, r') J_p(r') dr'$$

wherein (Fig. 1) $E_i(r)$ is the incident field on the domain $D$, $G$ the free space Green function [19]

$$G(r, r') = -\frac{k^2}{4\omega e_0} H_0^{(2)}(k|r - r'|)$$

$k$ is the free-space wave number, $\omega$ the angular central frequency, $e_0$ the free-space permittivity, and $H_0^{(2)}$ the Hankel function of order zero and second kind.

Furthermore, it is supposed that $\int_F |J_p(r')|^2 dr'$ is finite, namely, the radiated (real and imaginary) energy is finite [11].

Paralleling the discussion in [6], the kernel of the radiation operator is square integrable and consequently, $A_T$ is a compact (or completely continuous) operator [19, 20]. The field $E_i(r)$ can be represented by means of singular-value decomposition (SVD), also called Hilbert–Schmidt decomposition

$$E_i(r) = \sum_{n=1}^{\infty} \sigma_m(J_p, v_m) L_2(r) u_n(r)$$

wherein $(u_m, v_m, \sigma_m)$ are the left singular functions, the right singular functions and the singular values of the operator, respectively, and $\langle \cdot, \cdot \rangle_{L_2(r)}$ denotes the inner product in the space of square–integrable functions of support $F$. It is recalled that the null-space of the operator is the linear subspace consisting of all the non-radiating currents having support $F$ [21]. The left singular functions represent an orthonormal basis in the space of the radiating currents having support $F$, whereas the right singular functions represent an orthonormal basis for the range of the operator, namely, for the set of incident fields on the target. The singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq \ldots$ are an infinite countable set of real, positive numbers and $\lim_{n \to \infty} \sigma_n = 0$ because of the compactness of the operator.

Now, consider the operator $B$ relating the incident field $E_i$ to the current density $J_i$ induced on the target. At this point, the main question regards the properties of the range of this operator, and in particular if the set of currents $J_i$ induced on the target is bounded. In the case of a dielectric scattering object, a finite uniform norm of the contrast $\chi$ (wherein $\chi$ is the difference between the relative permittivities of the medium and free space) assures that $J_i$ belongs to a bounded set in $L_2$ under the natural hypothesis that the energy of the main sources is finite [15]. In the case of perfectly conducting surfaces, the norm of $J_i$ on the scattering object depends on the behaviour of the surface current density on the edges. In particular, the Meixner condition [22] assures that, excluding the unphysical case of infinitely thin edge, the current density is square integrable.

The operator current density induced on the target is related to the scattered field observed in $r_s \in F$ by the integral operator $A$, whose explicit expression is

$$E_s(r_s) = \int_D G(r_s, r') J_s(r') dr'$$

wherein $G$ is the Green function reported in (2).

The operator (4) is similar to the operator (1), but the domain and the range of the two operators are switched. Consequently, the field $E_s(r_s)$ on the observation manifold can be represented by means of SVD,

$$E_s(r_s) = \sum_{n=1}^{\infty} \sigma_m(J_s, u_m) L_2(r) v_n(r_s)$$

**Figure 1** Geometry of the problem

**Figure 2** RADAR signal propagation model
wherein \((u_\alpha, v_\alpha, \sigma_\alpha)\) is the singular system of the integral operator in (1), \(\ast\) denotes the conjugate and \(\langle \cdot, \cdot \rangle_{L^2(D)}\) denotes the inner product in the space of square-integrable functions of support \(D\).

Accordingly, the scattered field on \(D\) can be represented as [16]

\[
E_s(r) = \sum_{n=1}^{\infty} \sigma_n s^\ast_n \langle \mathbf{B}_0, u^n \rangle_{L^2(D)} v_n^\ast(r) = \sum_{m,n=1}^{\infty} \sigma_m \sigma_n s^\ast_m \langle \mathbf{B}_0, u^n \rangle_{L^2(D)} v_m^\ast v_n^\ast(r) \tag{6}
\]

The scattered field \(E_s\) is observed by means of a proper detector, in our case, a set of RX antennas, modelled as a measurement operator \(M\) which relates the incident field to the observable quantity \(z\). Note that, since in any practical case, the observable quantity is affected by noise and measurement uncertainties, \(M\) is a ‘noisy’ operator, in the sense that it introduces noise and uncertainties in the observable data. The presence of noise makes it impossible to measure \(E_s\) within an arbitrary precision, limiting the accuracy on \(E_s\). This observation will play a relevant role in the next section.

Note that the operator \(M\) depends on the details of antennas (distance, polarisation, gain etc.). For the sake of simplicity, \(M\) is often modelled as a spatial sampling operator, for example, an operator that samples the field \(E_s\) in the positions of the RX antennas, plus an additive noise.

Finally, the signal at the output of the measurement operator is processed by means of processing operator \(P\), and the output \(y\) is compared with a suitable threshold to detect the presence of a target. Although the most part of the research on SMR is devoted to \(P\), this paper is focused on the preceding blocks of the scheme shown in Fig. 2.

3 NDF of the scattered field in MA-RADAR

As a preliminary step, the concept of NDF of a system is recalled. Generally speaking, with reference to a physical system on which some a priori information are available, the NDF is the minimum number of parameters that enables to uniquely identify the state of the system within a given accuracy, which is generally fixed by the noise level and/or by the measurement accuracy. Since attention is restricted to linear representations, the NDF can be equivalently defined as the minimum number of linearly independent functions required to represent the system within the required accuracy. A rigorous discussion on the NDF of the field is reported in [11]. An intuitive approach is also reported in [12].

For the sake of clarity, as a first step, the NDF of the incident field \(E_i\) will be discussed. Consider the singular-value expansion of the incident field truncated to the first \(M\) terms. Because of the compactness of the radiation operator

\[
E_i(r) = \sum_{m=1}^{M} \sigma_m \langle f_m, \psi_m \rangle_{L^2(D)} u_m(r)
\]

\[
= \sum_{m=M+1}^{\infty} \sigma_m \langle f_m, \psi_m \rangle_{L^2(D)} u_m(r) \tag{7}
\]

Since the singular values of the operator tend to zero and \(J_p\) is bounded, (7) shows that it is possible to uniformly approximate \(E_i\) using a base consisting of a finite number of singular functions within any desired finite degree of accuracy. Furthermore, it is possible to demonstrate that the basis consisting of singular functions is also an optimal base, in the sense that no other base exists, allowing to obtain the same approximation accuracy with a smaller number of basis functions [23].

According to the definition of the NDF, the dimension of such an optimal basis is the NDF of the incident field at a fixed level of accuracy. Furthermore, it is possible to show that the singular values of the radiation operator have a step-like behaviour with a fast decrease after a knee, provided that the electromagnetic source is electrically large and the radiated field is observed at a distance of at least a couple of wavelengths from the sources [11, 16]. In this case, the sum in (3) has a number of elements only slightly larger than \(N_{knee}\) wherein \(N_{knee}\) is the number of singular values of the radiation operator before the knee [11, 6].

If the observation domain is very close (less than one wavelength) to the sources, the decrease of the singular values after the knee tends to be slow, and the NDF turns out significantly larger than \(N_{knee}\) (which becomes hardly identifiable) and strongly dependent on the noise level [16]. However, because of the wavelengths used in RADAR systems, this is not a case of interest in this paper.

According to the above discussion, and taking into account (3), it is noted that in any case in which a knee is clearly identifiable, the minimum number of linear functions to represent the scattered field on the observation manifold is scarcely dependent on the approximation level, and turns out to be practically equal to \(N_{knee}\). According to [11], this number will be called the NDF of the field without explicit reference to the approximation level and will be denoted in the following as NDF\(_F\), wherein the subscript \(F\) stands for field.

The above observations are a straightforward consequence of the compactness of the radiation operators (1) and (4). Since this property is true also in the 3D geometry, the discussion on the NDF can be straightforwardly extended to the 3D case. For example, in the 2D scalar case, a simple and effective approximation of the NDF, in case of electrically large scattering objects and observation domain encircling the source and placed at a distance not smaller than a couple of

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In this case, the incident field is expanded using one TX antenna and more than one RX antenna. In RADARs, that are usually defined as RADAR systems having more than one TX antenna and only one RX antenna, is the number of terms required to approximate the scattered field received by the antennas of a SIMO RADAR, wherein SM stands for SIMO. In the above case, the singular functions \( f_0(r) \) are fixed. Note that this case includes the SIMO RADARs, that are usually defined as RADAR systems using one TX antenna and more than one RX antenna. In this case, the incident field \( E_i \) on \( \Sigma \) is fixed, and only NDFF free parameters are in (6). This means that no more than NDFF parameters are required to reconstruct the spatial distribution of the signal received by the antennas of a SIMO RADAR. Consequently, the (spatial) NDF of the signal received by a SIMO RADAR system is limited by the NDFF in the following will be denoted as NDFSM, wherein SM stands for SIMO.

Exploiting the Lorentz reciprocity theorem, the (spatial) NDF of the signal received by a MISO RADAR, that uses more than one TX antenna and only one RX antenna, is again limited by the NDFF.

Now consider the MIMO RADAR systems. In this case, different incident fields can be chosen, exploiting the double sum. However, because of the Lorentz reciprocity theorem, \( E(r_{\alpha}, r'_{\beta}) = E(r_{\beta}, r'_{\alpha}) \). This symmetry limits the number of independent coefficients in the representation to only NDFF (NDFF + 1)/2 \( \cong N_{\text{knee}} (N_{\text{knee}} + 1)/2 \). Consequently, the NDF of the signal received by a MIMO RADAR is not greater than \( \cong N_{\text{knee}}/2 \) and is larger than NDFSM. This quantity will be called NDFMM, wherein MM stands for MIMO RADAR.

### 4 Bandlimited approximation of the scattered field

The singular system in (6) can be obtained in closed form only in a limited number of cases. One of such cases is a scatterer included in a circumference \( \Sigma \) having radius \( a \) and transmitters and receivers placed on a circumference \( C \) having radius \( R \), concentric to \( \Sigma \). In this case, it is possible to expand the incident and scattered fields by cylindrical harmonics, obtaining [15]

\[
E_S(\theta', \varphi') \cong \sum_{m,n=-N}^{N'} E_S(\theta_m, \varphi_n) D_N(\theta - \theta_m) D_N(\varphi' - \varphi_n)
\]  

(8)

wherein \( \theta \) is the angular observation position, \( \theta' \) the angular source position, \( 2N' + 1 \cong N_{\text{knee}} = 2Ba, \) \( \beta \) is the free-space wavenumber, \( D_N(\theta) = \sin[(2k + 1)\theta/2]/[(2k + 1)\sin(\theta/2)] \) is the Dirichlet function, \( \varphi_n = m\Delta\varphi, \varphi_n = n\Delta\varphi, \Delta\varphi = 2\pi/(2N' + 1) \). Note that (8) is a sampling representation whose coefficients are the values of the scattered field in points whose angular distance \( \Delta\varphi \) is fixed by the Nyquist step. Furthermore, \( E_S(\theta_m, \varphi_n') = E_S(\theta_n, \varphi_m') \) because of reciprocity.

In the above case, the singular functions \( \{v_n\} \) turn out to be bandlimited, and consequently, the scattered field is bandlimited as well. The use of bandlimited representations allows a straightforward use of the literature on the signal processing. In particular, it allows a simple evaluation of the autocorrelation function, which has a key role in the estimation of SMR performance, as will be shown in the next section. Consequently, it is of great interest to represent the scattered field by a bandlimited representation in more general cases. This is indeed possible considering the so-called ‘reduced field’ \( F(\xi) = E(\xi)\exp(i\omega\varphi \xi) [17, 18] \), obtained from the field by introducing a proper parameterisation \( \xi(s) \) (wherein \( s \) is the curvilinear abscissa along the observation curve) on the observation curve, and extracting a proper phase function \( \phi(\xi) \). For an introduction to the reduced field, the reader is invited to refer to [6, 17, 18]. It is only recalled that the reduced field can be approximated by a bandlimited function, whose bandwidth is slightly larger than a value \( W \) called effective bandwidth, within any degree of accuracy. After fixing the accuracy of the representation, the number of samples at Nyquist step required to represent the field using the reduced field is only slightly larger than the NDF of the field and is almost equal to \( 2\pi\Theta \), wherein \( \Theta \) is the length of the observation curve in \( \xi \) units, and \( \omega \) is the spatial frequency bandwidth (note that in [17, 18] an angular frequency bandwidth is used, whereas in this paper, a frequency spatial bandwidth is adopted to obtain a clearer parallelism between the signals in the spatial domain and the signals in the temporal domain). As discussed above, the value of \( \omega \) turns out to be slightly larger than the value of the effective bandwidth \( W \). In the case of targets whose electrical dimension is large, this difference is very small, and in practice, the NDF turns out to be practically equal to \( 2\pi\Theta \) [6, 18]. The sampling representation allows to estimate the NDF of the field using an intuitive graphical representation, as shown in [6].

Since the reduced field is a bandlimited function, it can be represented by Kotelnikov–Whittaker–Shannon series. If \( \xi \) and \( \varphi' \) are the variables associated to the observation point...
and source point, then \[ F(\xi, \xi') \equiv \sum_{m_{\text{E}}, n_{\text{E}}=-N'}^{N'} F(\xi_m, \xi'_m) f(w(\xi - \xi_m)) f(w(\xi' - \xi'_m)) \]

wherein \( \xi_m \) and \( \xi'_m \) are the sampling positions, placed at Nyquist distance, \( f(\cdot) \) is the sinc function in case of open observation curves, or Dirichlet function in case of closed curves, and \( w \) is the spatial bandwidth of the reduced field. As discussed above, the value of \( 2N' + 1 \) turns out to be only slightly larger than \( N_{\text{max}} \).

Accordingly, the field can be represented as \[ E(\xi, \xi') \equiv \sum_{m_{\text{E}}, n_{\text{E}}=-N'}^{N'} E(\xi_m, \xi'_m) f(w(\xi - \xi_m)) \times f(w(\xi' - \xi'_m)) e^{j(\phi(\xi') - \phi(\xi))} \]

\[ R_{\text{EE}}(\xi_0, \xi; \xi'_0, \xi') \equiv \sum_{m_{\text{E}}, n_{\text{E}}=-N'}^{N'} E[E(\xi_0, \xi'_0) E^*(\xi_0, \xi'_0)] \times f(w(\xi - \xi_0)) f(w(\xi' - \xi'_0)) \times e^{-j(\phi(\xi) - \phi(\xi'_0))} \]

\[ R_{\text{EE}}(\xi_0, \xi; \xi'_0, \xi') \equiv \sum_{m_{\text{E}}, n_{\text{E}}=-N'}^{N'} E[E(\xi_0, \xi'_0) E^*(\xi_0, \xi'_0)] \times f(w(\xi - \xi_0)) f(w(\xi' - \xi'_0)) \times e^{-j(\phi(\xi) - \phi(\xi'_0))} \]

5 Target detection and degrees of diversity

As recalled in the Introduction, in the case of slow fading, the improvement of the SNR is obtained in wireless communication systems by taking advantage of the decorrelation between the different 'spatial channels' obtainable between the TX and RX antennas in multiple antenna systems. This improvement is called 'diversity gain' \[ 5 \].

Statistical MA-RADARs use the same method to contrast the slow fading of the Swerling case 1. In order to identify the number of statistically low-correlated channels, in the following, the TX antennas are modelled as point sources.

As a first step, consider the autocorrelation of the scattered field on an observation curve \( C \)

\[ R_{\text{EE}}(r_0, r; r'_{0}, r') = E[E(\xi_0, \xi'_0) E^*(\xi_0, \xi'_0)] \]

wherein \( E \) is the expectation operating on the ensemble of scatterers contained in \( \Sigma \), \( r_0 \) and \( r \) the two observation points and \( r'_{0} \) and \( r' \) the two source points. Even if the autocorrelation function can be obtained from the Hilbert–Schmidt expansion, the sampling representation is more suitable for our purposes.

In particular, the autocorrelation from (10) can be evaluated. Without loss of generality, it is supposed that the observation point \( \xi_0 \) and the source point \( \xi'_0 \) coincide with sampling positions, obtaining

\[ R_{\text{EE}}(\xi_0, \xi; \xi'_0, \xi') \equiv \sum_{m_{\text{E}}, n_{\text{E}}=-N'}^{N'} E[E(\xi_0, \xi'_0) E^*(\xi_0, \xi'_0)] \times f(w(\xi - \xi_0)) f(w(\xi' - \xi'_0)) \times e^{-j(\phi(\xi) - \phi(\xi'_0))} \]

For the sake of simplicity, consider a SIMO RADAR as a first step. In this case, the \( \xi_j = \xi_j \) and consequently, taken into account that \( f(w(\xi'_0 - \xi'_0)) = \delta_{0,0} \) (wherein \( \delta_{b,h} \) is the Kronecker symbol, equal to 0 if \( k \neq b \) and equal to 1 if \( k = b \))

\[ R_{\text{EE}}(\xi_0, \xi; \xi'_0, \xi') \equiv \sum_{m_{\text{E}}, n_{\text{E}}=-N'}^{N'} E[E(\xi_0, \xi'_0) E^*(\xi_0, \xi'_0)] \times f(w(\xi - \xi_0)) e^{-j(\phi(\xi) - \phi(\xi'_0))} \]

The autocorrelation of the field on the observation curve \( C \) depends on the autocorrelation of the field in the sampling positions. By placing an RX antenna at each sampling position, uncorrelated (or practically low correlated) signals can be obtained. Accordingly, in a SIMO RADAR the maximum number of uncorrelated signals (and the amount of diversity which can be obtained from the scattered signal) is practically equal to the NDF_SM. For reciprocity, this is also the maximum amount of diversity which can be obtained from a MISO RADAR.

In the case of a MIMO RADAR system, the discussion outlined for MISO RADAR for each of the TX positions can be obtained. However, because of the symmetry imposed by reciprocity, the maximum amount of diversity can be obtained from the scattered signal is only NDF_MM.

If a limited extended area for TX/RX antennas is used, as often happens, a relatively small NDF_F is obtained. In this case, the NDF depends also on the SNR, and the difference between the NDF_F and \( N_{\text{max}} \) can be significant. In any case, the use of a distance between the antennas significantly smaller than the Nyquist distance evaluated considering the surface \( \Sigma \) limiting the ensemble of possible targets does not give significant advantage.

The above discussion clearly shows the role of the NDF of the field in RADAR performance. In particular, parallelizing the discussion on MIMO communication systems, a RADAR system that exploits only one degree of freedom gives only 'array gain' \[ 5 \], namely, the increase of the SNR because of the presence of multiple antennas, whereas a RADAR system exploiting more than one NDF of the incident or scattered field is able to take advantage of the 'diversity gain', namely, the variation of scattered signal strength in the space, since it allows to obtain significantly decorrelated signals. As a consequence, the 'very true' nature of a statistical RADAR system does not depend on the number of TX/RX antennas, but on the NDF that the RADAR exploits. Statistically, MISO RADAR systems exploit more than one NDF of the incident field but only one NDF of the scattered field, whereas statistically, SIMO RADAR systems exploit only one NDF of the incident field but more than one NDF of the scattered field to obtain diversity gain. Statistically, MIMO RADARs exploit only one NDF of both the incident and scattered fields, allowing a full use of the available spatial
resource to contrast the temporal slow fading of the signal with a spatial fast fading.

Finally, in spite of the large number of TX/RX antennas possibly used, any RADAR system that exploits only one degree of freedom of the incident and scattered fields can give only array gain and consequently, from the point of view of the spatial diversity, it works like a SISO RADAR.

As discussed in the Introduction, the understanding of the ‘true nature’ of a RADAR system is important, since the processing algorithms for spatially coherent signals (e.g. in SISO RADARs) and spatially incoherent signals (in statistical MIMO, SIMO and MISO RADARs) are different.

As a first example, consider the circular geometry in the inset of Fig. 3. The autocorrelation function of the scattered field on the observation curve can be evaluated from (8), obtaining

\[ R_{EE}(\theta_0; \theta, \theta_0) = E\{E_\theta(\theta_0, \theta_0)E_\theta^*(\theta, \theta')\} \approx \sum_{m,n=-N}^{N} \left\{ E(\theta_0, \theta_0)E^*(\theta_m, \theta_n) \right\} \times D_N(\theta - \theta_m)D_N(\theta' - \theta_n) \]  

wherein it is supposed that the points \( \theta_0 \) and \( \theta_0' \) coincide with sampling positions. All the scatterers are placed inside \( \Sigma \). The field is sampled considering the case of sources inside a circle of radius \( a \), obtaining an angular sampling distance \( \Delta \theta \approx 2\pi/(2b \alpha) \) [11]. As discussed above, this step assures that spatial information is not lost for any ensemble of targets that can fill \( \Sigma \).

In the following, examples regarding MIMO RADAR are presented.

The target is modelled as a large number (100) of scattering points randomly placed in a circle having radius \( a = 8 \lambda \).

A number of nine TX/RX RADAR antennas are placed along a circumference placed in the far field of the scattering object at a constant angular step. According to the incoherent nature of the signals received by a statistical MIMO radar [2, 3], the sum \( y = \sum_{m,n=1}^{9} |E_{\theta}(\theta_m, \theta_n)|^2 \) of the squared amplitude of the signals scattered by the target considering all the TX and RX combinations was evaluated. The quantity \( y \) is a random variable since it depends on the position of the scattering points with respect to the TX/RX antennas. It must be stressed that the data are noise free since the aim of the example is to show the dependence of the cumulative distribution function (CDF) of the random variable \( y \) on the angular distance between the TX/RX antennas.

Since the measured quantity is the square amplitude of the scattered field, the bandwidth of the signal doubles, and the Nyquist angular distance \( \Delta \theta \) is considered equal to \( \Delta \theta = \pi/(2N + 1) \). As discussed before, in this case, it must be expected that the NDF is slightly larger than \( \approx 4N \), so that there are low-correlated signals considering a distance a bit smaller than \( \Delta \theta = \pi/(2N + 1) \).

In Fig. 3, the CDF of normalised \( y \) is shown in the case of 2500 different randomly chosen positions of the 100 scattering points. The plot shows results obtained with nine TX/RX antennas very close to each other \((10^{-3} \Delta \theta, \) denoted as SISO RADAR), and at distance 0.5 \( \Delta \theta \) (e.g. two TX/RX antennas each Nyquist intervals), \( \Delta \theta \) (e.g. one TX/RX antenna each Nyquist intervals), 2 \( \Delta \theta \) and 4 \( \Delta \theta \).

For a given \( y \) value, the plot shows a fast decreasing of the CDF of \( y \) until the TX/RX antennas are placed at a distance not larger than a couple of \( \Delta \theta \), after which the improvement is slower, confirming the discussion on the NDF in MIMO statistical RADAR.

In more general geometries, the autocorrelation from (13) can be evaluated. In the following, a statistical MIMO RADAR is considered, in which the TX/RX antennas are placed along a straight observation line. The ensemble of targets is limited by an ellipsoidal curve \( \Sigma \). In this case, the sampling positions, namely, in our case the position of the TX/RX antennas, are placed in the intersection points between the observation line and a number of hyperbolas whose foci coincide with the foci of \( \Sigma \) ([16] and [18]), and consequently depend on the position of the target with respect to the area covered by the antennas. As an example, the CDF of \( y \) is plotted for two different positions of the target, respectively, above (Fig. 4) and far (Fig. 5) from the MIMO RADAR set of antennas. The ensemble of targets consists of 2500 different targets, simulated by 100 scattering points randomly placed in an ellipses having semi-axes 25 \( \lambda \) and 4 \( \lambda \) at an altitude of 1000 \( \lambda \). The nine TX/RX antennas are positioned at a constant distance each other, and the overall extension of the RADAR system covers \( t \) Nyquist intervals \( \Delta \xi \), with \( \Delta \xi \) varying from \( 9 \times 10^{-3} \Delta \xi \) (this case is indicated in the figures as SISO.

Figure 3 MIMO RADAR with a target placed in a circular domain and antennas placed on an arch
In the first example, regarding the straight observation line, the target is above the fifth antenna of the RADAR system, whereas in the second example, the antenna closest to the target is at a distance equal to 1700 l with respect to the vertical of the target. The results confirm the role of the Nyquist distance in collecting independent information. It is useful to stress that in the two examples, the same distance in terms of Nyquist intervals correspond to very different 'true' distances. In fact, the Nyquist interval is constant in the s variable, but not with respect to the curvilinear abscissa s along the observation line (see e.g. Fig. 3 in [6]). Coming back to the two examples, in the first case, the length of a RADAR system covering nine Nyquist intervals is 90 l (e.g. an average distance of 10 l between two successive antennas), whereas in the second case, the length of a RADAR system covering the same nine Nyquist intervals is almost 600 l (e.g. an average distance of almost 70 l between two successive antennas). This confirms that the area that must be covered by the antennas to collect independent information is almost constant as a function of the ξ parameter, but it is variable as a function of the curvilinear abscissa s and depends on the position of the target. These results are parallel to the results obtained in [6], namely, that the amount of independent information that can be obtained observing the field along a curve depends on the number of Nyquist intervals falling on the curve (e.g. the quantity in [6] is called the Nyquist length' of the curve), and suggests that other investigations must be undertaken to understand the true performance of statistical MIMO RADARs in particular in the practically relevant case of distant targets flying at low altitude.

6 Conclusions

This paper studies the performance of statistical RADAR systems based on multiple antennas using the concept of (spatial) NDF of the field scattered by the target.

Two main results are shown in the paper. The first one is that the NDF of the scattered field limits the performance of RADARs using multiple antennas. The second one is that the ’true nature’ of statistical SIMO, MISO and MIMO RADARs depends not on the number of TX and/or RX antennas, but on the NDF of the field that the RADAR system exploits. In particular, statistical SIMO, MISO and MIMO RADARs take advantage of more than one spatial degree of freedom to obtain diversity gain, whereas a MA-RADAR that uses only one degree of freedom works from the point of view of spatial diversity, as a SISO RADAR, since it allows only array gain.

Some examples regarding the statistical MIMO RADAR are discussed, confirming the role of the NDF of the field in the performance of such a RADAR system. Even if the model and the examples discussed in this paper are limited to a 2D geometry, the results can be easily extended to 3D full vector case.

The method adopted in this paper is strongly based on the parallelism between communication systems and RADAR systems. This suggests an investigation on the application of the many new communication techniques to MA-RADAR. As an example, it is useful to note that among the ‘spatial channels’ there is a high probability that a channel has a ‘gain’ higher than the average gain. In this case, it might be interesting to investigate a strategy similar to the so-called ‘opportunistic communications’ in wireless systems [5], in which all the power is sent through the channel having the higher gain. For example, consider a MIMO RADAR in which the total transmitted power is constrained by some stealth requirements. An unusually high false alarm probability is fixed. For a potential target, the TX and RX antennas are identified assuring the best SNR, and this couple of antennas are used. For a Swerling case 1, the ‘spatial-temporal channel’ connecting these two antennas is almost constant during two pulses (or equivalently the coherence time is higher than the temporal interval between two pulses), a high SNR can be obtained, assuring a high
detection probability with a low false alarm probability. This is just an example of an useful parallelism between RADAR systems and communication systems.

7 References


