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“NONLINEAR DYNAMICAL SYSTEMS AND CHAOS:
PHENOMENOLOGICAL AND COMPUTATIONAL ASPECTS”

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CHAOS IS ALL AROUND US.....

“THERE LIES A BEHAVIOR BETWEEN RIGID REGULARITY AND RANDOMNESS BASED ON PURE CHANCE: IT IS CALLED CHAOS”

HOW TO ACCOUNT FOR THE COMMONLY OBSERVED CHAOTIC BEHAVIOR BETWEEN THESE TWO EXTREMES?
WHEN ONE SEES IRREGULARITY IN ENGINEERING, PHYSICAL, BIOLOGICAL AND SYSTEMS FROM LIFE SCIENCES, ONE CLINGS TO RANDOMNESS AND DISORDER FOR EXPLANATIONS AND THE WHOLE VAST MACHINERY OF PROBABILITY AND STATISTICS IS APPLIED.

BUT RANDOMNESS IS NOT THE ONLY WAY.......
RATHER RECENTLY, IN FACT, IT HAS BEEN REALIZED THAT THE TOOLS OF CHAOS THEORY CAN BE APPLIED TO UNDERSTAND, MANIPULATE AND CONTROL A VARIETY OF SYSTEMS WITH MANY PRACTICAL APPLICATIONS.

TO UNDERSTAND WHY IT IS TRUE ONE MUST START ON A WORKING KNOWLEDGE OF HOW CHAOTIC SYSTEMS BEHAVE.
CHAOS VS RANDOMNESS

DO NOT CONFUSE CHAOS WITH RANDOMNESS

RANDOMNESS
irreproducible and unpredictable

CHAOS
searching for possible definitions
A DEFINITION FOR CHAOS?

AS FOR MANY TERMS IN SCIENCE, THERE IS NO STANDARD DEFINITION FOR CHAOS

THE TYPICAL FEATURES OF CHAOS INCLUDE:

NONLINEARITY

if it is linear....it cannot be chaotic

DETERMINISM

it has deterministic (rather than probabilistic) underlying rules, known initial conditions...every future state of the system must follow
SENSITIVITY TO INITIAL CONDITIONS
small changes in its initial state can lead to radically different behavior in its final state. (butterfly effect)

SUSTAINED IRREGULARITY
hidden order including a large or infinite number of unstable periodic patterns: this hidden order forms the infrastructure of irregular chaotic systems

LONG TERM PREDICTIONS MOSTLY IMPOSSIBLE
due to sensitivity to initial conditions which can be known only to a finite degree of precision
CHAOS IN NATURE

CHAOS IS WIDELY PRESENT IN NATURE THROUGH A VARIETY OF SYSTEMS AND IT EVEN SEEMS THAT NATURE MAY TREASURE AND EXPLOIT CHAOS.

DOES NATURE EXPLOIT CHAOS?

NORMAL BRAIN ACTIVITY MAY BE CHAOTIC AND PATHOLOGICAL ORDER MAY INDEED BE THE CAUSES OF SUCH DISEASES AS EPILEPSY
DOES NATURE EXPLOIT CHAOS?

It has been speculated that too much periodicity in heart rates might indicate disease.

Perhaps the chaotic characteristic of human body are better adapted to its chaotic environment.

It is even suspected that biological systems exploit chaos to store, encode and decode informations.
WHERE HAS CHAOS COME FROM?

Historically the study of chaos started in mathematics and physics, to be then expanded into engineering, biology, life sciences and, more recently, into information and social sciences.

Recently it has been developed a considerable interest in commercial and industrial applications of chaotic systems.

The key of this interest is in the power of easily accessible computer hardwares, for speed and memory capacity.
A BRIEF HISTORY OF CHAOS

1887

KING OSCAR II OF SWEDEN ANNOUNCED A PRIZE FOR THE FIRST PERSON WHO COULD SOLVE THE n-BODY PROBLEM TO DETERMINE THE ORBITS OF n-CELESTIAL BODIES AND THUS ANSWER THE QUESTION

"IS THE SOLAR SYSTEM STABLE?"
Henri Poincaré, a French mathematician, won the first prize in King’s Oscar contest by being the closest to solve the n-body problem with his work on the three body problem.

Newton solved the 2-body problem

Poincarè showed that the 3-body problem is essentially “unsolvable”
POINCARE', CONSIDERED, FOR EXAMPLE, JUST THE SUN, EARTH AND MOON ORBITING IN A PLANE UNDER THEIR MUTUAL GRAVITATIONAL ATTRACTIONS.

LIKE THE PENDULUM, THE SYSTEM HAS UNSTABLE SOLUTIONS.

INTRODUCING A POINCARE' SECTION, HE FOUND THAT OMOCLINIC TANGLES MUST OCCUR, DISCOVERING THAT THE ORBIT OF THREE OR MORE CELESTIAL BODIES CAN EXHIBIT UNSTABLE AND UNPREDICTABLE BEHAVIOR.

THUS CHAOS IS BORN.....EVEN IF NOT YET NAMED!!
EDWARD LORENZ showed irregularity in a toy model for weather forecasting. Using a computer, he discovered the first chaotic attractor:

\[
\begin{align*}
x' &= -s(x-y) \\
y' &= r x - y - xz \\
z' &= xy - bz
\end{align*}
\]

Changing in time these variables, gives a trajectory in a 3D space.
SOME VERY QUALITATIVE REMARKS:

THIS IS A VERY SIMPLE SYSTEM OF EQUATIONS WITH DISSIPATION

FROM ALL STARTS, TRAJECTORIES SETTLE ONTO A STRANGE, CHAOTIC ATTRACTOR.

SINCE THE SOLUTION IS CHAOTIC, IT CANNOT BE WRITTEN DOWN IN ANY FORMULA. IN A MATHEMATICAL SENSE, THE PROBLEM IS "UNSOLVABLE"

THIS COULD NOT BE DISCOVERED WITHOUT THE COMPUTERS THAT APPEARED IN THE 1960's. ALL THE COMPUTERS DO, IS SOLVE THE PROBLEM IN AN APPROXIMATE WAY
TIEN-LI AND JAMES YORKE, IN THE PAPER

"PERIOD THREE IMPLIES CHAOS"

MADE USE FOR THE FIRST TIME OF THE TERM

"CHAOS THEORY"
ROBERT M. MAY, APPLIED THE LOGISTIC EQUATION TO ECOLOGY, SHOWING THE OCCURRENCE OF CHAOTIC BEHAVIOR IN POPULATION DYNAMICS.
MITCHELL FEIGENBAUM, FOUND
"UNIVERSAL FEATURES" ASSOCIATED
WITH THE WAY SYSTEMS APPROACH
TO CHAOS
BENOIT MANDELBROT, SHOWED FRACTAL GEOMETRY WITH APPLICATIONS TO COMPUTER GRAPHICS AND IMAGE COMPRESSION (*)
1990

**EDD OTT, CELSO GREBOGI AND JAMES YORKE**, GIVE BIRTH TO THE CHAOS CONTROL THEORY

1990

**LOU PECORA**, INTRODUCED THE IDEA OF SYNCHRONIZATION OF CHAOTIC SYSTEMS
HOW APPLYING CHAOS?

THERE IS A GREAT NUMBER OF POTENTIAL COMMERCIAL AND INDUSTRIAL APPLICATIONS, BASED ON DIFFERENT ASPECTS OF CHAOTIC SYSTEMS

ACCORDING TO SUCH ASPECTS, APPLICATION TYPES CAN BE CLASSIFIED INTO THREE MAIN CATEGORIES:

STABILIZATION AND CONTROL

SYNTHESIS

ANALYSIS
STABILIZATION AND CONTROL

The extreme sensitivity of chaotic systems to tiny perturbations can be manipulated to stabilize or control systems.

In fact, tiny perturbations can be artificially incorporated either to keep a large system stable (stabilization) or to direct a large chaotic system into a desired state (control).
STABILIZATION AND CONTROL

CAREFULLY CHOSEN CHAOS CONTROL INTERVENTIONS MIGHT LEAD TO MORE EFFICIENT

✓ AIRPLANE WINGS
✓ TURBINES
✓ CHEMICAL REACTIONS IN INDUSTRIAL PLANTS
✓ IMPLANTABLE DEFIBRILLATORS
✓ BRAIN PACEMAKERS
✓ COMPUTERS NETWORKS
STABILIZATION AND CONTROL

THE UNDERLYING PRINCIPLE TO CONTROL AND STABILIZE CHAOTIC SYSTEMS BY USING THEIR EXTREME SENSITIVIES HAVE BEEN IN PRACTICE EXPLOITED THROUGH

✓ NASA SATELLITE CONTROL
✓ MULTIMODE LASER
✓ BELOUSOV-ZHABOTINSKY CHEMICAL REACTIONS
✓ HEARTH ARRYTMIAS
SYNTHESIS OF CHAOTIC SYSTEMS

Artificially generated chaotic systems may be applied to certain types of problems to make the systems, whether chaotic or not chaotic, work better.

In fact

Depending on the type of problem, regularity is not always the best. It might be advantageous to violate one of the classical dogmas - linearity - to design devices using, rather than avoiding, non linearity and chaos.
SYNTHESIS OF Chaotic Systems

In applications, many linear systems to do one thing well, can in practice be replaced with fewer and more flexible nonlinear systems that exploit chaos

✓ Biological neural networks

✓ Synchronized chaos

✓ Encryption

✓ Chaos in communications
ANALYSIS AND PREDICTION OF CHAOTIC SYSTEMS

THE STUDY OF CHAOS MAY LEAD TO BETTER DETECTION AND PREDICTION ALGORITHMS FOR CHAOTIC SYSTEMS.

IN FACT

AMAZINGLY, THE LACK OF LONG-TERM PREDICTABILITY IN CHAOTIC SYSTEMS DOES NOT IMPLY THAT SHORT TERM PREDICTION IS IMPOSSIBLE. DIFFERENTLY FROM PURELY RANDOM SYSTEMS, CHAOTIC SYSTEMS CAN BE PREDICTED FOR A SHORT TERM IN THE FUTURE.
ANALYSIS AND PREDICTION OF CHAOTIC SYSTEMS

THE BASIC IDEA IS THAT A SUCCESSFUL ANALYSIS OF THE TIME SERIES OF A CHAOTIC SYSTEM, ALLOW PREDICTION OR FORECASTING OF THE SYSTEM’S BEHAVIOR IN THE NEAR FUTURE. SUCH ANALYSIS IS IN GENERAL ENOUGH DIFFICULT AND MUCH MORE WORK MAY BE NEEDED FOR MASSIVE APPLICATIONS

PREDICTION PROBLEMS....

✓ CHAOTIC DISEASES
✓ CARDIOLOGY
✓ ECOLOGY
✓ WEATHER AND CLIMATE
# Potential Applications of Types of Chaos

<table>
<thead>
<tr>
<th>Category</th>
<th>Applications</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>First application of chaos is control of irregular behavior in devices and systems</td>
</tr>
<tr>
<td>Synthesis</td>
<td>Potential control of epilepsy, improved dithering of systems as lasers, computers networks</td>
</tr>
<tr>
<td>Information Processing</td>
<td>Encoding, decoding and storage of information in chaotic systems; better performance of neural networks; patterns recognition</td>
</tr>
<tr>
<td>Short-Term Prediction</td>
<td>Contagious diseases, weather, economy</td>
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# Potential Application Areas of Chaos

<table>
<thead>
<tr>
<th>Category</th>
<th>Applications</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Engineering</strong></td>
<td>vibration control; stabilization of circuits; chemical reactions; turbines; lasers; combustion; power grids; etc.</td>
</tr>
<tr>
<td><strong>Computers</strong></td>
<td>computer networks; encryption; control of chaos in robotic networks; etc.</td>
</tr>
<tr>
<td><strong>Communications</strong></td>
<td>information compression and storage; computer networks design and management; etc.</td>
</tr>
<tr>
<td><strong>Biomedicine</strong></td>
<td>cardiology; heart rhythm (EEG) analysis; prediction and control of irregular heart activity</td>
</tr>
<tr>
<td><strong>Management and Finance</strong></td>
<td>economic forecasting; financial analysis; market prediction and intervention</td>
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CHAOS IN EXPERIMENTAL AND THEORETICAL MODELS

As widely shown, chaos is observed in many models, experimental as well as theoretical.

Theoretical models from real life are often expressed, according to the nature of the case under study, through nonlinear dynamical systems which can be “qualitatively” defined as deterministic “prescription” for evolving the state of the system forward in time.
You can think of a dynamical system as the time evolution of some physical system, like the motion of a few planets under the influence of their respective gravitational forces. Usually you want to know the fate of system for long times, like, will the planets eventually collide or will the system persist for all times? For some systems (e.g., just two planets) these questions are relatively simple to answer since it turns out that the motion of the system is regular and converges (e.g.) to an equilibrium.

However, many interesting systems are not that regular! In fact, it turns out that for many systems even very close initial conditions might get spread far apart in short times. For example, you probably have heard about the motion of a butterfly which can produce a perturbation of the atmosphere resulting in a thunderstorm a few weeks later.
A dynamical system is a semigroup $G$ acting on a space $M$. That is, there is a map

$$T : G \times M \rightarrow M$$

$$\quad (g, x) \mapsto T_g(x)$$

(such that)

$$T_g \circ T_h = T_{gh}.$$  \hspace{1cm} (6.2)

If $G$ is a group, we will speak of an invertible dynamical system.

We are mainly interested in discrete dynamical systems where

$$G = \mathbb{N}_0 \quad \text{or} \quad G = \mathbb{Z}$$

(6.3)

and in continuous dynamical systems where

$$G = \mathbb{R}^+ \quad \text{or} \quad G = \mathbb{R}.$$  \hspace{1cm} (6.4)

The prototypical example of a discrete dynamical system is an iterated map. Let $f$ map an interval $I$ into itself and consider

$$T^n = f^n = f \circ f^{n-1} = f \circ \cdots \circ f, \quad G = \mathbb{N}_0.$$  \hspace{1cm} (6.5)

Clearly, if $f$ is invertible, so is the dynamical system if we extend this definition for $n = \mathbb{Z}$ in the usual way. You might suspect that such a system is too simple to be of any interest. However, we will see that the contrary is the case and that such simple system bear a rich mathematical structure with lots of unresolved problems.

The prototypical example of a continuous dynamical system is the flow of an autonomous differential equation

$$T_t = \Phi_t, \quad G = \mathbb{R},$$  \hspace{1cm} (6.6)
CONTINUOUS DYNAMICAL SYSTEMS

Now we will have a closer look at the solutions of an autonomous system

\[ \dot{x} = f(x), \quad x(0) = x_0. \] (6.7)

Throughout this section we will assume \( f \in C^k(M, \mathbb{R}^n), \ k \geq 1 \), where \( M \) is an open subset of \( \mathbb{R}^n \).

Such a system can be regarded as a vector field on \( \mathbb{R}^n \). Solutions are curves in \( M \subseteq \mathbb{R}^n \) which are tangent to this vector field at each point. Hence to get a geometric idea of how the solutions look like, we can simply plot the corresponding vector field.

In particular, solutions of the IVP (6.7) are also called integral curves or trajectories. We will say that \( \phi \) is an integral curve at \( x_0 \) if it satisfies \( \phi(0) = x_0 \).

As in the previous chapter, there is a (unique) maximal integral curve \( \phi_x \) at every point \( x \), defined on a maximal interval \( I_x = (T_-(x), T_+(x)) \).

Introducing the set

\[ W = \bigcup_{x \in M} I_x \times \{x\} \subseteq \mathbb{R} \times M \] (6.8)

we define the flow of our differential equation to be the map

\[ \Phi : W \to M, \quad (t, x) \mapsto \phi(t, x), \] (6.9)

where \( \phi(t, x) \) is the maximal integral curve at \( x \). We will sometimes also use \( \Phi_x(t) = \Phi(t, x) \) and \( \Phi_t(x) = \Phi(t, x) \).
CONTINUOUS DYNAMICAL SYSTEMS

EQUILIBRIA

The orbit of $x$ is defined as

$$\gamma(x) = \Phi(I_x, x) \subseteq M.$$  \hfill (6.17)

Note that $y \in \gamma(x)$ implies $y = \Phi(t, x)$ and hence $\gamma(x) = \gamma(y)$ by (6.12). In particular, different orbits are disjoint (i.e., we have the following equivalence relation on $M$: $x \simeq y$ if $\gamma(x) = \gamma(y)$). If $\gamma(x) = \{x\}$, then $x$ is called a fixed point (also singular, stationary, or equilibrium point) of $\Phi$. Otherwise $x$ is called regular and $\Phi(\cdot, x) : I_x \hookrightarrow M$ is an immersion.

Similarly we introduce the forward and backward orbits

$$\gamma_\pm(x) = \Phi((0, I_\pm(x)), x).$$ \hfill (6.18)

PERIODIC ORBITS

Clearly $\gamma(x) = \gamma_-(x) \cup \{x\} \cup \gamma_+(x)$. One says that $x \in M$ is a periodic point of $\Phi$ if there is some $T > 0$ such that $\Phi(T, x) = x$. The lower bound of such $T$ is called the period, $T(x)$ of $x$, that is, $T(x) = \inf\{T > 0| \Phi(T, x) = x\}$. By continuity of $\Phi$ we have $\Phi(T(x), x) = x$ and by the flow property $\Phi(t + T(x), x) = \Phi(t, x)$. In particular, an orbit is called periodic orbit if one (and hence all) point of the orbit is periodic.

It is not hard to see (Problem 6.7) that $x$ is periodic if and only if $\gamma_+(x) \cap \gamma_-(x) \neq \emptyset$ and hence periodic orbits are also called closed orbits.
CONTINUOUS DYNAMICAL SYSTEMS

WHAT TO TAKE ACCOUNT FOR?

EQUILIBRIA

PERIODIC ORBITS

Hence we may classify the orbits of \( f \) as follows:

(i) fixed orbits (corresponding to a periodic point with period zero)
(ii) regular periodic orbits (corresponding to a periodic point with positive period)
(iii) non-closed orbits (not corresponding to a periodic point)
CONTINUOUS DYNAMICAL SYSTEMS

PROPERTIES

Hence we may classify the orbits of $f$ as follows:

(i) fixed orbits (corresponding to a periodic point with period zero)
(ii) regular periodic orbits (corresponding to a periodic point with positive period)
(iii) non-closed orbits (not corresponding to a periodic point)

$x \in M$ is called $\sigma$ complete, $\sigma \in \{\pm\}$, if $T_\sigma(x) = \sigma \infty$ and complete if it is both $+$ and $-$ complete (i.e., if $I_x = \mathbb{R}$).

Lemma 2.10 gives us a useful criterion when a point $x \in M$ is $\sigma$ complete.

**Lemma 6.3.** Let $x \in M$ and suppose that the forward (resp. backward) orbit lies in a compact subset $C$ of $M$. Then $x$ is $+$ (resp. $-$) complete.

Clearly a periodic point is complete. If all points are complete, the vector field is called complete. Thus $f$ complete means that $\Phi$ is globally defined, that is, $W = \mathbb{R} \times M$. 
A set $U \subseteq M$ is called $\sigma$ invariant, $\sigma \in \{\pm\}$, if
\[ \gamma_\sigma(x) \subseteq U, \quad \forall x \in U, \] (6.19)
and invariant if it is both $\pm$ invariant, that is, if $\gamma(x) \subseteq U$. If $U$ is $\sigma$ invariant, the same is true for $\overline{U}$, the closure of $U$. In fact, $x \in \overline{U}$ implies the existence of a sequence $x_n \in U$ with $x_n \to x$. Fix $t \in I_x$. Then (since $W$ is open) for $N$ sufficiently large we have $t_n \in I_{x_n}$, $n \geq N$ and
\[ \Phi(t, x) = \lim_{n \to \infty} \Phi(t_n, x_n) \in \overline{U}. \]

Clearly, arbitrary intersections and unions of $\sigma$ invariant sets are $\sigma$ invariant. Moreover, the closure of a $\sigma$ invariant set is again $\sigma$ invariant.

Clearly, arbitrary intersections and unions of $\sigma$ invariant sets are $\sigma$ invariant. Moreover, the closure of a $\sigma$ invariant set is again $\sigma$ invariant.

If $C \subseteq M$ is a compact $\sigma$ invariant subspace, then Lemma 6.3 implies that all points in $C$ are $\sigma$ complete.

A nonempty, compact, $\sigma$ invariant set is called minimal if it contains no proper $\sigma$ invariant subset possessing these three properties.

**Lemma 6.4.** Every nonempty, compact ($\sigma$) invariant set $C \subseteq M$ contains a minimal ($\sigma$) invariant set.

**Lemma 6.5.** Every $\sigma$ invariant set $C \subseteq M$ homeomorphic to an $m$-dimensional disc (where $m$ is not necessarily the dimension of $M$) contains a singular point.
The $\omega_\pm$-limit set of a point $x \in M$, $\omega_\pm(x)$ is the set of those points $y \in M$ for which there exists a sequence $t_n \to \pm \infty$ with $\Phi(t_n, x) \to y$.

Clearly, $\omega_\pm(x)$ is empty unless $x$ is $\pm$ complete. Observe, that $\omega_\pm(x) = \omega_\pm(y)$ if $y \in \gamma(x)$ (if $y = \Phi(t, x)$ we have $\Phi(t_n, y) = \Phi(t_n, \Phi(t, x)) = \Phi(t_n + t, x)$). Moreover, $\omega_\pm(x)$ is closed. Indeed, if $y \notin \omega_\pm(x)$ there is a neighborhood $U$ of $y$ disjoint from $\Phi(\{t \in I_x|t > T\}, x)$ for some $T > 0$. Hence the complement of $\omega_\pm(x)$ is open.

The set $\omega_\pm(x)$ is invariant since if $\Phi(t_n, x) \to y$ we have

$$\Phi(t_n + t, x) = \Phi(t, \Phi(t_n, x)) \to \Phi(t, y)$$

(6.21)

for $|t_n|$ large enough since $x$ is $\pm$ complete.

In summary,

Lemma 6.6. The set $\omega_\pm(x)$ is a closed invariant set.
CONTINUOUS DYNAMICAL SYSTEMS

PROPERTIES

Problem 6.7 (Periodic points). Let $\Phi$ be the flow of some differential equation.

(i) Show that if $T$ satisfies $\Phi(T, x) = x$, the same is true for any integer multiple of $T$. Moreover, show that we must have $T = nT(x)$ for some $n \in \mathbb{Z}$ if $T(x) \neq 0$.

(ii) Show that a point $x$ is stationary if and only if $T(x) = 0$.

(iii) Show that $x$ is periodic if and only if $\gamma_+(x) \cap \gamma_-(x) \neq \emptyset$ in which case $\gamma_+(x) = \gamma_-(x)$ and $\Phi(t + T(x), x) = \Phi(t, x)$ for all $t \in \mathbb{R}$. In particular, the period is the same for all points in the same orbit.

Problem 6.8. A point $x \in M$ is called nonwandering if for every neighborhood $U$ of $x$ there is a sequence of positive times $t_n \to \infty$ such that $\Phi_{t_n}(U) \cap U \neq \emptyset$ for all $t_n$. The set of nonwandering points is denoted by $\Omega(f)$.

(i) $\Omega(f)$ is a closed invariant set (Hint: show that it is the complement of an open set).

(ii) $\Omega(f)$ contains all periodic orbits (including all fixed points).

(iii) $\omega_+(x) \subseteq \Omega(f)$ for all $x \in M$. 
CONTINUOUS DYNAMICAL SYSTEMS

ATTRACTING SET

An invariant set $\Lambda$ is called attracting if there exists some neighborhood $U$ of $\Lambda$ such that $U$ is positively invariant and $\Phi_t(x) \rightarrow \Lambda$ as $t \rightarrow \infty$ for all $x \in U$. The sets

$$W^\pm(\Lambda) = \{x \in M| \lim_{t \rightarrow \pm\infty} d(\Phi_t(x), \Lambda) = 0\}$$

(9.8)

are the stable respectively unstable sets of $\Lambda$. Here $d(A, B) = \inf\{||x - y||| x \in A, y \in B\}$ denotes the distance between two sets $A, B \subseteq \mathbb{R}^n$. The set $W^+(\Lambda)$ is also called the domain or basin of attraction for $\Lambda$. It is not hard to see that we have

$$W^+(\Lambda) = \bigcup_{t<0} \Phi_t(U) = \{x \in M| \omega_+(x) \subseteq \Lambda\}.$$  

(9.9)

But how can we find such a set? Fortunately, using our considerations from above, there is an easy way of doing so. An open connected set $E$ whose closure is compact is called a trapping region for the flow if $\Phi_t(E) \subseteq E$, $t > 0$. In this case

$$\Lambda = \omega_+(E) = \bigcap_{t \geq 0} \Phi(t, E)$$

(9.10)

is an attracting set by construction.

PROPERTIES

In fact, an attracting set will always contain the unstable manifolds of all its points.

**Lemma 9.4.** Let $E$ be a trapping region, then

$$W^-(x) \subseteq \omega_+(E), \quad \forall x \in \omega_+(E).$$

(9.11)
CONTINUOUS DYNAMICAL SYSTEMS

To exclude such situations, we can define an attractor to be an attracting set which is topologically transitive. Here a closed invariant set $\Lambda$ is called topologically transitive if for any two open sets $U, V \subseteq \Lambda$ there is some $t \in \mathbb{R}$ such that $\Phi(t, U) \cap V \neq \emptyset$. In particular, an attractor cannot be split into smaller attracting sets. Note that $\Lambda$ is topologically transitive if it contains a dense orbit (Problem 9.2).

As another example let us look at the Duffing equation

$$\ddot{x} = -\delta \dot{x} + x - x^3, \quad \delta \geq 0,$$

(9.12)

from Problem 7.4. It has a sink at $(-1, 0)$, a hyperbolic saddle at $(0, 0)$, and a sink at $(1, 0)$. The basin of attraction of the sink $(-1, 0)$ is bounded by the stable and unstable manifolds of the hyperbolic saddle $(0, 0)$. The situation for $\delta = 0.3$ is depicted below.

Finally, let us consider the van der Pol equation (8.26). The unique periodic orbit is an attractor and its basin of attraction is $\mathbb{R}^2 \setminus \{0\}$. However, not all attractors are fixed points or periodic orbits, as the example in our next section will show.

Problem 9.1. Show that $\omega_+(X)$ is invariant under the flow.

Problem 9.2. Show that a closed invariant set which has a dense orbit is topologically transitive.
"SIMPLE" ATTRACTORS...

FIXED POINTS

PERIODIC ORBITS
....AND MORE "COMPLEX" ATTRACTORS

TORUS

STRANGE ATTRACTOR
CONTINUOUS DYNAMICAL SYSTEMS: STABILITY OF EQUILIBRIA

As already mentioned earlier, one of the key questions is the long time behavior of the dynamical system (6.7). In particular, one often wants to know whether the solution is stable or not. But first we need to define what we mean by stability. Usually one looks at a fixed point and wants to know what happens if one starts close to it. Hence we define the following.

DEFINITIONS

A fixed point $x_0$ of $f(x)$ is called stable if for any given neighborhood $U(x_0)$ there exists another neighborhood $V(x_0) \subseteq U(x_0)$ such that any solution starting in $V(x_0)$ remains in $U(x_0)$ for all $t \geq 0$.

Similarly, a fixed point $x_0$ of $f(x)$ is called asymptotically stable if it is stable and if there is a neighborhood $U(x_0)$ such that

$$\lim_{t \to \infty} |\phi(t, x) - x_0| = 0 \quad \text{for all } x \in U(x_0).$$

(6.30)

STABILITY VIA DEFINITION
CONTINUOUS DYNAMICAL SYSTEMS

STABILITY VIA LIAPUNOV FUNCTIONS

Pick a fixed point $x_0$ of $f$ and an open neighborhood $U(x_0)$ of $x_0$. A Liapunov function is a continuous function

$$L : U(x_0) \to \mathbb{R}$$

(6.36)

which is zero at $x_0$, positive for $x \neq x_0$, and satisfies

$$L(\phi(t_0)) \geq L(\phi(t_1)), \quad t_0 < t_1, \quad \phi(t) \in U(x_0) \{x_0\},$$

(6.37)

for any solution $\phi(t)$. It is called a strict Liapunov function if equality in (6.37) never occurs. Note that $U(x_0) \{x_0\}$ can contain no periodic orbits if $L$ is strict (why?).

Since the function $L$ is decreasing along integral curves, we expect the level sets of $L$ to be positively invariant. Let $S_0$ be the connected component of $\{x \in U(x_0)|L(x) \leq \delta\}$ containing $x_0$. First of all note that

Theorem 6.11 (Liapunov). Suppose $x_0$ is a fixed point of $f$. If there is a Liapunov function $L$, then $x_0$ is stable. If, in addition, $L$ is not constant on any orbit lying entirely in $U(x_0) \{x_0\}$, then $x_0$ is asymptotically stable. This is for example the case if $L$ is a strict Liapunov function.

Most Liapunov functions will in fact be differentiable. In this case (6.37) holds if and only if

$$\frac{d}{dt}L(\phi(t, x)) = \nabla L(\phi(t, x)) \dot{\phi}(t, x) = \nabla L(\phi(t, x))f(\phi(t, x)) \leq 0.$$

(6.40)
Our aim in this chapter is to show that a lot of information of the stability of a flow near a fixed point can be read off by linearizing the system around the fixed point. But first we need to discuss stability of linear autonomous systems

\[ \dot{x} = Ax. \quad (7.1) \]

**Theorem 7.1.** Denote the eigenvalues of \( A \) by \( \alpha_j \), \( 1 \leq j \leq m \), and the corresponding algebraic and geometric multiplicities by \( a_j \) and \( g_j \), respectively.

The system \( \dot{x} = Ax \) is globally stable if and only if \( \text{Re}(\alpha_j) \leq 0 \) and \( a_j = g_j \) whenever \( \text{Re}(\alpha_j) = 0 \).

The system \( \dot{x} = Ax \) is globally asymptotically stable if and only if we have \( \text{Re}(\alpha_j) < 0 \) for all \( j \). Moreover, in this case there is a constant \( C \) for every \( \alpha < \min\{ -\text{Re}(\alpha_j) \}_{j=1}^m \) such that

\[ \| \exp(tA) \| \leq Ce^{-t\alpha}. \quad (7.4) \]
Theorem 7.2. The linear stable and unstable manifolds $E^\pm$ are invariant under the flow and every point starting in $E^\pm$ converges exponentially to 0 as $t \to \pm \infty$. In fact, we have

$$|\exp(tA)x| \leq Ce^{\mp t \alpha}|x|, \quad \pm t \geq 0, \quad x \in E^\pm, \quad (7.5)$$

for any $\alpha < \min\{|\text{Re}(\alpha)| | \alpha \in \sigma(A), \pm \text{Re}(\alpha) > 0\}$ and some $C > 0$ depending on $\alpha$.

In this section we want to transfer some of our results of the previous section to nonlinear equations. We define the stable, unstable set of a fixed point $x_0$ as the set of all points converging to $x_0$ for $t \to \infty$, $t \to -\infty$, that is,

$$W^\pm(x_0) = \{x \in M | \lim_{t\to \pm \infty} |\Phi(t, x) - x_0| = 0\}. \quad (7.6)$$

Both sets are obviously invariant under the flow. Our goal in this section is to find these sets.

We define the stable respectively unstable manifolds of a fixed point $x_0$ to be the set of all points which converge exponentially to $x_0$ as $t \to \infty$ respectively $t \to -\infty$, that is,

$$M^\pm(x_0) = \{x \in M \sup_{t \geq 0} e^{\pm \alpha t} |\Phi(t, x) - x_0| < \infty \text{ for some } \alpha > 0\}. \quad (7.8)$$

Both sets are invariant under the flow by construction.
Theorem 7.11 (Hartman-Grobman). Suppose \( f \) is a differentiable vector field with 0 as a hyperbolic fixed point. Denote by \( \Phi(t, x) \) the corresponding flow and by \( A = df_0 \) the Jacobian of \( f \) at 0. Then there is a homeomorphism \( \varphi(x) = x + h(x) \) with \( h \) bounded such that
\[
\varphi \circ e^{tA} = \Phi_t \circ \varphi
\] (7.34)
in a sufficiently small neighborhood of 0.

Two systems with vector fields \( f, g \) and respective flows \( \Phi_f, \Phi_g \) are said to be topologically conjugate if there is a homeomorphism \( \varphi \) such that
\[
\varphi \circ \Phi_{f, t} = \Phi_{g, t} \circ \varphi
\] (7.40)
Note that topological conjugacy of flows is an equivalence relation.

The Hartman-Grobman theorem hence states that \( f \) is locally conjugate to its linearization \( A \) at a hyperbolic fixed point. In fact, there is even a stronger results which says that two vector fields are locally conjugate near hyperbolic fixed points if and only if the dimensions of the stable and unstable subspaces coincide.
But before that, let me point out that it is also interesting to look at the change of a differential equation with respect to a parameter $\mu$. By Theorem 2.8 the flow depends smoothly on the parameter $\mu$ (if $\dot{f}$ does). Nevertheless very small changes in the parameters can produce large changes in the qualitative behavior of solutions. The systematic study of these phenomena is known as bifurcation theory. I do not want to go into further details at this point but I will rather show you some prototypical examples.

The system
\[ \dot{x} = \mu x - x^3 \]  \hspace{1cm} (6.33)
has one stable fixed point for $\mu \leq 0$ which becomes unstable and splits off two stable fixed points at $\mu = 0$. This is known as pitchfork bifurcation. The system
\[ \dot{x} = \mu x - x^2 \]  \hspace{1cm} (6.34)
has two stable fixed point for $\mu \neq 0$ which collide and exchange stability at $\mu = 0$. This is known as transcritical bifurcation. The system
\[ \dot{x} = \mu + x^2 \]  \hspace{1cm} (6.35)
has two stable fixed point for $\mu < 0$ which collide at $\mu = 0$ and vanish. This is known as saddle-node bifurcation.
CONTINUOUS DYNAMICAL SYSTEMS:
THE STUDY OF PERIODIC SOLUTIONS

THE CASE $n=2$ AND THE POINCARE'-BENDIXSON THEORY

Theorem 8.8 (Poincaré-Bendixson). Let $M$ be an open subset of $\mathbb{R}^2$ and $f \in C^1(M, \mathbb{R}^2)$. Fix $x \in M$, $\sigma \in \{\pm\}$, and suppose $\omega_\sigma(x) \neq \emptyset$ is compact, connected, and contains only finitely many fixed points. Then one of the following cases holds:

(i) $\omega_\sigma(x)$ is a fixed orbit.

(ii) $\omega_\sigma(x)$ is a regular periodic orbit.

(iii) $\omega_\sigma(x)$ consists of (finitely many) fixed points $\{x_j\}$ and unique non-closed orbits $\gamma(y)$ such that $\omega_\pm(y) \in \{x_j\}$.

Problem 8.4 (Bendixson’s criterion). Suppose $\text{div}\, f$ does not change sign and does not vanish identically in a simply connected region $U \subseteq M$. Show that there are no regular periodic orbits contained (entirely) inside $U$. (Hint: Suppose there is one and consider the line integral of $f$ along this curve. Recall the Gauss theorem in $\mathbb{R}^2$.)
CONTINUOUS DYNAMICAL SYSTEMS: 
THE STUDY OF PERIODIC SOLUTIONS

THE CASE n=2 AND THE POINCARE'-BENDIXON THEORY

Problem 8.4 (Bendixson’s criterion). Suppose \( \text{div } f \) does not change sign and does not vanish identically in a simply connected region \( U \subseteq M \). Show that there are no regular periodic orbits contained (entirely) inside \( U \). (Hint: Suppose there is one and consider the line integral of \( f \) along this curve. Recall the Gauss theorem in \( \mathbb{R}^2 \).)

Problem 8.5 (Dulac’s criterion). Show the following generalization of Bendixson’s criterion. Suppose there is a scalar function \( \alpha(x) \) such that \( \text{div}(\alpha f) \) does not change sign and does not vanish identically in a simply connected region \( U \subseteq M \), then there are no regular periodic orbits contained (entirely) inside \( U \).
CONTINUOUS DYNAMICAL SYSTEMS: 
THE STUDY OF PERIODIC SOLUTIONS

THE CASE n=2 AND THE POINCARE’-BENDIXON THEORY

One consequence of the Poincaré-Bendixon theory is that complicated dynamics are excluded in $\mathbb{R}^2$.

To have such kind of behavior, in the continuous case, one has to consider at least $\mathbb{R}^3$.

However, we will show through some very famous examples how even very simple system in $\mathbb{R}^3$ can exhibit a very complicated dynamics.
**CONDITIONS NECESSARY FOR CHAOS**

- System has at least three state variables
- The equations of motions are nonlinear

Therefore, in the continuous case......

\[ N \geq 3 \]

Along with nonlinearity means that chaos becomes possible!
Why Nonlinearity and 3D Phase Space?

There is no crossing in the phase space: so, how do complex motions arise?

By divergence, folding and mixing......possible with nonlinearity and at least a three dimensional phase space
WHAT CAN BE SAID WITHOUT SOLVING EQUATIONS?

✓ equilibria
✓ stability properties
✓ attractors
✓ Poincaré sections
Forget About Solving Equations!

THE LANGUAGE OF CHAOS

- Attractors (Dissipative Chaos)
- KAM torus (Hamiltonian Chaos)
- Poincare sections
- Lyapunov exponents and Kolmogorov entropy
- Fourier spectrum and autocorrelation functions
We end this section with an important lemma. Recall that a set \( \Sigma \subset \mathbb{R}^n \) is called a submanifold of codimension one (i.e., its dimension is \( n - 1 \)), if it can be written as
\[
\Sigma = \{ x \in U | S(x) = 0 \}, \tag{6.26}
\]
where \( U \subset \mathbb{R}^n \) is open, \( S \in C^k(U) \), and \( \partial S/\partial x \neq 0 \) for all \( x \in \Sigma \). The submanifold \( \Sigma \) is said to be transversal to the vector field \( f \) if \( (\partial S/\partial x)f(x) \neq 0 \) for all \( x \in \Sigma \).

**Lemma 6.8.** Suppose \( x \in M \) and \( T \in I_x \). Let \( \Sigma \) be submanifold of codimension one transversal to \( f \) such that \( \Phi(T, x) \in \Sigma \). Then there exists a neighborhood \( U \) of \( x \) and \( \tau \in C^k(U) \) such that \( \tau(x) = T \) and
\[
\Phi(\tau(y), y) \in \Sigma \tag{6.27}
\]
for all \( y \in U \).

**Proof.** Consider the equation \( S(\Phi(t, y)) = 0 \) which holds for \( (T, x) \). Since
\[
\frac{\partial}{\partial t} S(\Phi(t, y)) = \frac{\partial S}{\partial x}(\Phi(t, y)) f(\Phi(t, y)) \neq 0 \tag{6.28}
\]
for \( (t, y) \) in a neighborhood \( I \times U \) of \( (T, x) \) by transversality. So by the implicit function theorem (maybe after restricting \( U \)), there exists a function \( \tau \in C^k(U) \) such that for all \( y \in U \) we have \( S(\Phi(\tau(y), y)) = 0 \), that is, \( \Phi(\tau(y), y) \in \Sigma \). \( \square \)

If \( x \) is periodic and \( T = T(x) \), then
\[
P_{\Sigma}(y) = \Phi(\tau(y), y) \tag{6.29}
\]
is called Poincaré map.
CONTINUOUS DYNAMICAL SYSTEMS

- The original system flows in continuous time
- Poincarè section cuts across the phase-space orbits: on the section, one observes steps in discrete time
- The flow is replaced by an iterated map
- The dimension of the phase-space is reduced by one
DISCRETE DYNAMICAL SYSTEMS

WHAT TO TAKE ACCOUNT FOR?

FIXED POINTS

K-CYCLES

STABILITY PROPERTIES

BIFURCATIONS
Now let us introduce some notation for later use. To set the stage let $M$ be a metric space and let $f : M \to M$ be continuous. We are interested in investigating the dynamical system corresponding to the iterates

$$f^n(x) = f^{n-1}(f(x)), \quad f^0(x) = x. \quad (10.10)$$

In most cases $M$ will just be a subset of $\mathbb{R}^n$, however, the more abstract setting chosen here will turn out useful later on.

A point $p \in M$ satisfying

$$f(p) = p \quad (10.11)$$

is called a fixed point of $f$. Similarly, a fixed point of $f^n$,

$$f^n(p) = p, \quad (10.12)$$

is called a periodic point of period $n$. We will usually assume that $n$ is the prime period of $p$, that is, we have $f^m(p) \neq p$ for all $1 \leq m < n$.

The forward orbit of $x$ is defined as

$$\gamma_+(x) = \{f^n(x)|n \in \mathbb{N}_0\}. \quad (10.13)$$

It is clearly positively invariant, that is, $f(\gamma_+(x)) \subseteq \gamma_+(x)$. An orbit for $x$ is a set of points

$$\gamma(x) = \{x_n|n \in \mathbb{Z} \text{ such that } x_0 = x, \ x_{n+1} = f(x_n)\}. \quad (10.14)$$

It is important to observe that the points $x_{-n}, \ n \in \mathbb{N}$, are not uniquely defined unless $f$ is one to one. Moreover, there might be no such points at all (if $f^{-1}(x) = \emptyset$ for some $x_n$). An orbit is invariant, that is, $f(\gamma(x)) = \gamma(x)$. The points $x_n \in \gamma(x)$ are also called a past history of $x$. 
If $p$ is periodic with period $n$, then $\gamma_+(p)$ is finite and consists of precisely $n$ points

$$\gamma_+(p) = \{p, f(p), \ldots, f^{n-1}(x)\}. \quad (10.15)$$

The converse is not true since a point might be eventually periodic (fixed), that is, it might be that $f^k(x)$ is periodic (fixed) for some $k$.

For example, if $M = \mathbb{R}$ and $f = 0$, then $p = 0$ is the only fixed point and every other point is eventually fixed.

A point $x \in M$ is called forward asymptotic to a periodic point $p$ of period $n$ if

$$\lim_{k \to \infty} f^{nk}(x) = p. \quad (10.16)$$

The stable set $W^+(p)$ is the set of all $x \in M$ for which (10.16) holds. Clearly, if $p_1$, $p_2$ are distinct periodic points, their stable sets are disjoint. In fact, if $x \in W^+(p_1) \cap W^+(p_2)$ we would have $\lim_{k \to \infty} f^{n_1 n_2 k}(x) = p_1 = p_2$, a contradiction. We call $p$ attracting if there is an open neighborhood $U$ of $p$ such that $U \subseteq W^+(p)$. The set $W^+(p)$ is clearly positively invariant (it is even invariant $f(W^+(p)) = W^+(p)$ if $f$ is invertible).

Similarly, a point $x \in M$ is called backward asymptotic to a periodic point $p$ of period $n$ if there is a past history $x_n$ of $x$ such that $\lim_{k \to \infty} x_{-nk}(x) = p$. The unstable set $W^-(p)$ is the set of all $x \in M$ for which this condition holds. Again unstable sets of distinct periodic points are disjoint. We call $p$ repelling if there is an open neighborhood $U$ of $p$ such that $U \subseteq W^-(p)$. 
Theorem 10.1. Suppose $f \in C^1(U, U)$, $U \subset \mathbb{R}^n$, then a periodic point $p$ with period $n$ is attracting if all eigenvalues of $d(f^n)_p$ are inside the unit circle and repelling if all eigenvalues are outside.

If none of the eigenvalues of $d(f^n)$ at a periodic point $p$ lies on the unit circle, then $p$ is called hyperbolic. Note that by the chain rule the derivative is given by

$$d(f^n)(p) = \prod_{x \in \gamma_+(p)} df_x = df_{f^{n-1}(p)} \cdots df_f(p)df_p. \quad (10.17)$$

Finally, stability of a periodic point can be defined as in the case of differential equations. A periodic orbit $\gamma_+(p)$ of $f(x)$ is called stable if for any given neighborhood $U(\gamma_+(p))$ there exists another neighborhood $V(\gamma_+(p)) \subseteq U(\gamma_+(p))$ such that any point in $V(\gamma_+(p))$ remains in $U(\gamma_+(p))$ under all iterations. Note that this is equivalent to the fact that for any given neighborhood $U(p)$ there exists another neighborhood $V(p) \subseteq U(p)$ such that any point in $x \in V(p)$ satisfies $f^{nm}(x) \in U(p)$ for all $m \in \mathbb{N}_0$.

Similarly, a periodic orbit $\gamma_+(p)$ of $f(x)$ is called asymptotically stable if it is stable and attracting.
One of the most important and useful connections is in the qualitative study of complex systems.

**The Idea**

One way to make a complex system easier to analyze is by reducing the system to a simpler system that captures the important features of the original system.

For example in a continuous dynamical system, a Poincarè map (also called first-return map), living in a lower dimensional manifold, can provide much insight to the system.
In other words, a lower dimensional discrete time dynamical system is constructed from the original system.

Since the theory of one-dimensional maps is well developed, it will be useful if an appropriate one-dimensional map can be constructed from the system under study.